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Decidability vs. undecidability.
Logico-philosophico-historical remarks

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Abstract

The aim of the paper is to present the decidability problems from a philosophical and historical perspective as well as to indicate basic mathematical and logical results concerning (un)decidability of particular theories and problems.

1. Origin of the decidability problem

It was David Hilbert with who one should associate the beginning of research on the decidability – he drew attention of mathematicians to this problem and made it into a central problem of mathematical logic. He called it *das Entscheidungsproblem* (what literally means: "the decision problem").[1] It appeared in a sense already in his famous lecture at the Congress of Mathematicians in Paris in August 1900. Hilbert proposed there a list of 23 most important problems of mathematics which should be solved in the future. Problem X was (cf. [1]):

> Given a diophantine equation with any number of unknown quantities and with rational integral numerical coefficients: *To devise a process according to which it can be determined by a finite number of operations whether the equation is solvable in rational integers*.[2]

In the first quarter of the 20th century Hilbert formulated and developed a research programme, called today Hilbert programme. Its aim was the

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[1] It seems that the word *das Entscheidungsproblem* appeared for the first time in the paper "Das Entscheidungsproblem der mathematischen Logik" of H. Behmann given at the meeting of Deutsche Mathematiker-Vereiningung in Göttingen in May 1921 – cf. [1], page 21.

[2] Eine diophantische Gleichung mit irgendwelchen Unbekannten und mit ganzen rationalen Zahlkoeffizienten sei vorgelegt: *man soll ein Verfahren angeben, nach welchem sich mittels einer endlichen Anzahl von Operationen entscheiden läßt, ob die Gleichung in ganzen rationalen Zahlen lösbar ist.*

justification of the classical mathematics. In this context there appeared the decidability problem (closely connected with the completeness problem), that is the problem of finding an effective method which would enable us to decide in a finite number of prescribed steps whether a given formula is a theorem of a considered (formalized) theory. The first – not quite clear and explicite – formulation of this problem can be found in Hilbert's paper *Axiomatisches Denken* (1918; cf. [3]).[3] The decidability problem was formulated in a direct and explicite way by H. Behmann in his *Habilitationsschrift* in 1922 (cf. [6]) where he wrote:

> A well defined general procedure should be given which in the case of any given statement formulated with the help of purely logical means, would enable us to decide in a finite number of steps whether it is true or false or at least this aim would be realized within the – precisely fixed – framework in which its realization is really possible.[4]

Hilbert and Ackermann formulated the decidability problem in the book *Grundzüge der theoretischen Logik* in Chapter "Das Entscheidungsproblem im Funktionalkalkül und seine Bedeutung" in the following way (cf. [7], p. 73):

> The *Entscheidungsproblem* is solved when we know a procedure that allows, for any given logical expression, to decide by finitely many operations its validity or satisfiability.[5]

They distinguished there various aspects of *Entscheidungsproblem*:

- the *satisfiability problem* (or the *consistency problem*): given a formula, decide if it is consistent,
- the *validity problem*: given a formula, decide if it is valid,
- the *provability problem*: given a formula, decide if it is provable (in a given system).

In the first-order logic all those aspects are equivalent – this follows from the completeness theorem of Gödel. Moreover, by Deduction Theorem in the case of finitely axiomatizable theories it suffices to investigate only the system of the first-order logic and not theories in general (in the case of non-finitely axiomatizable theories it does not suffice). It should be added that the decision problem for a system of the first-order logic is called the *classical decision problem*.

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[3]One should admit that similar problems can be also found by Schröder in [4] and by Löwenheim in [5].

[4]Es soll eine ganz bestimmte allgemeine Vorschrift angegeben werden, die über die Richtigkeit oder Falschheit einer beliebig vorgelegten mit rein logischen Mitteln dargestellbaren Behauptung nach einer endlichen Anzahl von Schritten zu entscheiden gestattet, oder zum mindesten dieses Ziel innerhalb derjenigen – genau festzulegenden – Grenzen verwirklicht werden, innerhalb deren seine Verwirklichung tatsächlich möglich ist.

[5]Das Entscheidungsproblem ist gelöst, wenn man ein Verfahren kennt, das bei einem vorgelegten logischen Ausdruck durch endlich viele Operationen die Entscheidung über die Allegemeingültigkeit bzw. Erfülbarkeit erlaubt.

Hilbert and Ackermann were of the opinion that the *Entscheidungsproblem* is the main problem of mathematical logic. In [7] they wrote:

> The *Entscheidungsproblem* must be considered the main problem of mathematical logic. […] The solution of the *Entscheidungsproblem* is [an issue] of the fundamental significance for the theory of all domains whose propositions could be developed on the basis of a finite number of axioms.[6]

This conviction was based just on Deduction Theorem – indeed, a decision procedure for the first-order logic would generate (via this theorem) a decision procedure for any (finitely axiomatizable) first-order theory.

Other logicians shared the conviction of the importance of the *Entscheidungsproblem*. P. Bernays and M. Schönfinkel wrote in [8]:

> The central problem of mathematical logic, which is also most closely related to the question of axiomatics, is the *Entscheidungsproblem*.[7]

J. Herbrand begins the paper [9] with the words:

> We could consider the fundamental problem of mathematics to be the following. Problem A: What is the necessary and sufficient condition for a theorem to be true in a given theory having only finite number of hypotheses?

Herbrand finished the paper [10] with the words:

> The solution of this problem [i.e., the decision problem – R.M.] would yield a general method in mathematics and would enable mathematical logic to play with respect to classical mathematics the role that analytic geometry plays with respect to ordinary geometry.[8]

In [11] Herbrand added:

> In a sense it is [the classical decision problem – R.M.] the most general problem of mathematics.

F.P. Ramsey wrote in [12] (p. 264) that this paper was:

> concerned with a special case of one of the leading problems in mathematical logic, the problem of finding a regular procedure to determine the truth or falsity of any given logical formula.

The roots of the decision problem can be traced while back by those philosophers who were interested in a general method of problem solving. One should mention here first of all the medieval thinker Raimundus Lullus and his *ars magna* as well as Descartes' idea of *mathesis universalis* and Gottfried Wilhelm Leibniz with his *characteristica universalis* and *calculus ratiocinator*.

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[6] Das Entscheidungsproblem muss als das Hauptproblem der mathematischen Logik bezeichnet werden. […] Die Lösung des Entscheidungsproblems ist für die Theorie aller Gebiete, deren Sätze überhaupt einer logischen Entwickelbarkeit aus endlich vielen Axiomen fähig sind, von grundsätzlicher Wichtigkeit.

[7] Das zentrale Problem der mathematischen Logik, welches auch mit den Fragen der Axiomatik im engsten Zusammenhang steht, ist das Entscheidungsproblem.

[8] La solution de ce problème fournirait une méthode générale en Mathématique, et permettrait de faire jouer à la logique mathématique, vis-à-vis de Mathématique classique, le même rôle que la géométrie analytique vis-à-vis de la géométrie ordinaire.

The realization of the latter idea should provide a method of mechanical solving of any scientific problem (expressed in the symbolic language). Partial realization of this idea (in fact restricted to mathematics) was found by the mathematical logic at the end of the 19th and the beginning of the 20th century.

One should note that Leibniz distinguished between two different versions of *ars magna*:

- *ars inveniendi* which finds all true scientific statements,
- *ars iudicandi* which allows one to decide whether any given scientific statement is true or not.

In fact, in the framework of the first-order logic an *ars inveniendi* exists: the collection of all valid first-order formulas is recursively enumerable, hence there is an algorithm that lists all valid formulas. On the other hand, the classical decision problem can be viewed as the *ars iudicandi* problem in the first-order framework. It can be formulated in the following way: Does there exist an algorithm that decides the validity of any given first-order formula?

It is worth noticing that some logicians felt sceptical about the possibility of finding such an algorithm. Among them was J. von Neumann who wrote in [13] (pp. 11–12):

> It appears thus that there is no way of finding the general criterion for deciding whether or not a well-formed formula *a* is provable. (We cannot, however, at the moment demonstrate this. Indeed, we have no clue as to how such a proof of undecidability would go.) […] The undecidability is even *the condition sine qua non* for the contemporary practice of mathematics, using as it does heuristic methods, to make any sense. The very day on which the undecidability would cease to exist, so would mathematics as we now understand it; it would be replaced by an absolutely mechanical prescription, by means of which anyone could decide the provability or unprovability of any given sentence.
>
> Thus we have to take the position; it is generally undecidable, whether a given well-formed formula is provable or not. The only thing we can do is […] to construct an arbitrary number of provable formulas. […] In this way, we can establish for many well-formed formulas that they are provable. But in this way, we never succeed to establish that a well-formed formula is not provable. [9]

[9] Es scheint also, dass es keinen Weg gibt, um das allgemeine Entscheidungskriterium dafür, ob eine gegebene Normalformel *a* beweisbar ist, aufzufinden. (Nachweisen können wir freilich gegenwärtig nichts. Es ist auch gar kein Anhaltspunkt dafür vorhanden, wie ein solcher Unentscheidbarkeitsbeweis zu führen wäre.) [...] Und die Unentscheidbarkeit ist sogar die *Condition sine qua non* dafür , dass es überhaupt einen Sinn habe, mit den heutigen heuristischen Methoden Mathematik zu treiben. An dem Tage, an dem die Unentscheidbarkeit aufhörte, würde auch die Mathematik im heutigen Sinne aufhören zu existieren; an ihre Stelle würde eine absolut mechnische Vorschrift treten, mit deren Hilfe jedermann von jeder gegebenen Aussage entscheiden könnte, ob diese beweisen werden kann oder nicht.

J. Herbrand in an appendix to his [11] wrote:

> Note finally that, although at present it seems unlikely that the decision problem can be solved, it has not yet been proved that it is impossible to do so.

2. First (negative) results

Note that before the 1930s some positive answers to the decision problem for particular theories have been obtained (we shall say more about those results later). However, the classical decision problem (i.e., the decision problem for the first-order logic) was unsolved. Notice also that to prove that there is no effective procedure to decide the formulas of the first-order logic one needs a precise definition of the notion of an algorithm and of an effective method (in the case of positive solutions one does not need a precise general definition). In fact, in the 1930s such definitions have been given by Church, Gödel, Turing, Herbrand, Kleene. The Church-Turing thesis formulated in 1936 stated that those precise definitions are adequate with respect to the intuitive notion of an effective procedure.[10]

The method of an arithmetization of syntax introduced by Gödel enabled also to formulate precisely the decision problem. This was done by Alfred Tarski in [15]. Tarski proposed the following definitions:

– a (first-order) theory *T* is said to be *decidable* if and only if the set of (Gödel numbers of) theorems of *T* is recursive,
– a (first-order) theory *T* is said to be *undecidable* if and only if the set of (Gödel numbers of) theorems of *T* is not recursive,
– a (first-order) theory *T* is said to be *essentially undecidable* if and only if *T* is undecidable and every consistent extension *T'* of T (in the same language as *T*) is undecidable.

The first result concerning the *Entscheidungsproblem* in a strict formulation was the theorem by A. Church in 1936 (cf. [16]) providing the negative solution of the decision problem for the first-order predicate calculus. In fact Church has proved that the set of all valid formulas of the first-order logic is not effectively decidable. A similar result was obtained a bit later by A. Turing. The method used was similar in both cases: it was shown that a certain undecidable combinatorial problem can be represented in the first-order logic, hence the latter is undecidable. In fact, Church has shown that the set of provable formulas (theorems) of the first-order logic is not λ-definable. The undecidability of the first-order logic is then a corollary via the completeness theorem (due to Gödel, 1929) and the Church-Turing thesis.

The result of Church has been later "sharpened", i.e., it has been shown that the *Entscheidungsproblem* has a negative solution for a fragment of the first-

[10] On Church-Turing thesis, its history and epistemological status see [14].

order logic, viz. for the first-order predicate calculus in a language containing at least one binary predicate – this was done by Kalmár [17]. This contrasted with the earlier result by L. Löwenheim (1915), Th. Skolem (1919) and H. Behmann (1922) on the decidability of the classical monadic first-order predicate calculus. Note that the intuitionistic monadic first-order predicate calculus is not decidable!

The undecidability result of Church implied also undecidability of the second-order logic (add that – as noticed above – Hilbert and Ackermann were talking about logic as such not distinguishing the order of it). Today one knows that the first-order logic being undecidable is semi-decidable, i.e., the set of (Gödel numbers of) its theorems is recursively enumerable whereas the second-order logic is not even semi-decidable (this follows from Gödel's incompleteness theorem).

3. Studies on (un)decidability

Church's result was beginning of very intensive studies on the problem of (un)decidability. For a long time those problems were treated as central in the mathematical logic and the foundations of mathematics. Together with investigations on the complexity of decision procedures a new group of problems appeared, viz. the problem of how complicated the possible decision procedures can be.

To systematize the presentation of the results obtained let us distinguish studies on the (un)decidability of: (a) concrete mathematical theories, (b) fragments of the first-order logic (both are connected via Deduction Theorem), and (c) problems in the computation theory. Those investigations contributed to the development of both mathematical logic and the recursion theory. The literature is very extensive here. In what follows examples of most important results in the indicated fields will be provided and some remarks on the methods used in the proofs will be given.

3.1. (Un)decidability of mathematical theories

Investigations on the decidability of mathematical theories were carried out long before the theorem of Church – in fact to prove the decidability of a theory one does not need a precise definition of decidability itself (only for a negative result such a definition is necessary!).

The main methods used (nowadays) in proving the decidability of (mathematical) theories are the following:[11]

– elimination of quantifiers,

[11]Detailed information on the methods used in proving the decidability or the undecidability of theories together with examples and references to the literature can be found, e.g., in [18].

– modeltheoretic method,
– method of interpretation.

The first two methods are based in a theorem due to A. Janiczak (1950) and stating that if a theory T is consistent, complete and (recursively) axiomatizable then T is decidable. Methods (i) and (ii) are used just to show that a considered theory is complete and to obtain in this way its decidability.

The method of elimination of quantifiers consists of indicating a set of a certain class Φ of formulas in the language of the considered theory (called basic formulas) such that (a) every formula of Φ is decidable, (b) any formula of T is T-equivalent to a Boolean combination of some formulas form Φ and (c) the decidability of basic formulas implies the decidability of the Boolean combinations of them.

The method was initiated by L. Löwenheim (1915) and used in the fully-developed form by Th. Skolem (1919) and C.H. Langford (1927). It was also intensively studied at the seminar conducted by A. Tarski at Warsaw University in 1927–29. Tarski and his students used this method to characterize definability and to prove the decidability of particular mathematical theories (cf. [19]). It was also used to describe and classify all complete extensions of a given theory. The elimination of quantifiers became there *the* method and a paradigm of how logicians should study axiomatic theories. The very name of the method comes from Tarski.

With the help of the method of elimination of quantifiers, the decidability of various theories has been established, in particular the following theories have been shown to be decidable:

– the arithmetic of addition (Presburger arithmetic) (Presburger, 1928–1930),
– elementary theory of identity (Löwenheim, 1915),
– theory of finitely many sets (Löwenheim, 1915),
– theory of discrete order DO (Langford, 1927),
– theory of linear order in the set of rationals (Tarski, 1936),
– theory of algebraically closed fields ACF (1949),
– theory of Boolean algebras (Tarski, 1949),
– theory of real numbers (Tarski, 1951; Cohen, 1969),
– theory of abelian groups (Szmielew, 1949),
– theory of well order (Tarski, Mostowski, Donner, 1949, 1978).[12]

Note that Presburger arithmetic T_+ is the first-order theory in the language $L(T_+)$ with 0, S, + as the non-logical constants and based on the following non-logical axioms:

$$0 \neq S(x),$$

[12]On the (dramatic) history of this result see [19].

$$S(x) = S(y) \rightarrow x = y,$$
$$x + 0 = x,$$
$$x + S(y) = S(x + y),$$

nduction scheme for formulas φ of the language L(T_+).

Notice that results on the decidability of the first-order theory of successor (Herbrand, 1928) and of the theory of multiplication (with successor but without addition!) (Skolem, 1930) has been also obtained. Note that in contrast to those results the arithmetic of successor, addition and multiplication is essentially undecidable!

The second indicated method, i.e., the modeltheoretic method is usually used to show (by methods of model theory) that a given theory is complete or to study systematically its all complete extensions. Sometimes a combination of this method and the method of elimination of quantifiers is used. Using those methods one has proved the decidability of the following theories:

- theory of linear dense order without the first and last element DNO,
- theory of algebraically closed fields of a given characteristic,
- theory of *p*-adic fields (Ax, Kochen, 1965-1966),
- theory of all finite fields (Ax, 1968),
- theory of real closed fields RCF,
- theory of linearly ordered sets (Ehrenfeucht, 1959 – result announced only; Läuchli 1966).

The last indicated method of proving decidability, i.e., the method of interpretation can be briefly described as follows. Let a decidable theory T_0 formalized in a language L_0 be given. We are asking if another given theory T formalized in a language L is decidable. To answer this question one defines a computable (recursive) function *f* mapping formulas of the language L on formulas of the language L_0 such that if φ is a sentence of L then *f*(φ) is a sentence of L_0 and $T \vdash \varphi$ if and only if $T_0 \vdash f(\varphi)$. This gives us a decision procedure for the theory T.

By this method the decidability of, e.g., the second-order monadic theory of one successor S1S (Büchi, 1962) and of the weak second-order theory of one successor WS1S (Büchi, 1960; Elgot, 1961) have been proved.

In the case of decidable theories one can ask the question: how complex is a decision procedure? To indicate some answers and to show that decision procedures are usually very complicated (mostly of exponential complexity) and hence not applicable practically, let us mention the following results:

- A decision procedure for Presburger arithmetic is of the complexity at least $2^{2^{cn}}$ for a certain constant $c>0$, i.e. to decide whether a formula φ of the length *n* is a theorem of Presburger arithmetic one needs at least $2^{2^{cn}}$ steps [Fisher and Rabin, 1974].

- The complexity of the theory of multiplication is at least $2^{2^{2^{cn}}}$ for a certain constant $c>0$ [Fisher and Rabin, 1974].

The problem whether there are decidable theories with practically applicable decision procedures is still open. It is connected with the famous problem of whether $\mathcal{P}=\mathcal{NP}$, which is nowadays the central problem of the recursion theory and of the complexity theory.

Let us turn now to proofs of the undecidability. Main methods of proving the undecidability of a theory are the following:

i) the method based directly on the ideas of Gödel's proof of the incompleteness theorem,
ii) the method of interpretation.

The method (i) is based on the theorem stating that if all recursive relations are strongly representable in a theory T then T is undecidable (moreover, the set of Gödel numbers of theorems of T and the set of Gödel numbers of negations of theorems of T are not recursively separable). This method can be applied only in the case of theories which have built-in an appropriate fragment of the arithmetic of natural numbers built in.

The method (ii) has been mostly developed by A. Tarski. Generally speaking it consists in showing that a known undecidable theory T_1 can be interpreted (embedded) into a theory T_2 under question. If it is so then the theory T_2 is undecidable. Here are some examples of undecidable theories:

- Peano arithmetic (i.e., the arithmetic of natural numbers in the language with 0, S, +, · as non-logical constants),
- theory of rings (Tarski, 1951),
- theory of ordered fields (Robinson, 1949),
- theory of lattices (Tarski, 1951),
- predicate calculus with at least one binary predicate (Kalmár, 1936),
- theory of partial order (Tarski, 1951),
- theory of two equivalence relations whose intersection is the identity relation, theory of two equivalence relations, theory of one equivalence relation and one bijection (Janiczak, 1953),
- theory of groups (Tarski, 1953),
- theory of rationals with + and (J. Robinson, 1949).

The tenth problem of Hilbert mentioned at the very beginning of this history on (un)decidability was solved in the 1970s by Y. Matiyasevich who using some earlier results of M. Davis, H. Putnam and J. Robinson showed that a relation R is recursively enumerable if and only if R is diophantine. Since there are recursively enumerable relations which are not recursive, it follows that not every diophantine relation is recursive, hence the tenth problem of Hilbert has a negative solution, i.e., there is no effective method of deciding whether a given diophantine equation has solutions or not.

3.2. (Un)decidability of fragments of the first-order logic

Since the first-order predicate calculus is undecidable (as shown by Church), one can ask whether given fragments of it are decidable or not. This problem, called the classical decision problem, has been studied intensively. Nowadays this field of problems can be treated as closed (cf. the monograph [20]).

The investigated fragments of the first-order logic are usually described with the help of prefixes. Let us explain this using an example: so $\left[\forall\exists\forall,(\omega,1),(0)\right]$ denotes the class of all formulas in the prenex form with the quantifier prefix $\forall\exists\forall$ in the language with infinitely many unary predicates, one binary predicate and no function symbols. The symbol $\left[\forall\exists\forall,(\omega,1),(0)\right]_=$ denotes a similar class of formulas but now in the language there is the identity = symbol.

As an example of results obtained in studies of (un)decidability of fragments of the first-order logic let us say that the following fragments are undecidable:[13]

– $\left[\exists\forall\exists\forall,(0,3),(0)\right]$ (Büchi, 1962),
– $\left[\forall\exists\forall,(0,\infty),(0)\right]$ (Kahr, Moore, Wang, 1962),
– $\left[\forall^3\exists,(0,\infty),(0)\right]$ (Gödel, 1933),
– $\left[\exists^\infty\forall^2\exists^2\forall^\infty,(0,1),(0)\right]$ (Kalmár, 1932).

3.3. (Un)decidability in the computation theory

Problems of decidability have been studied also with respect to the computation theory. The best known result is here the undecidability of the halting problem. The question is: can it be effectively decided whether a given Turing machine stops at a given input x? The problem can be reformulated in the following way: let (φ_x) be the effective enumeration of all recursive functions. We are asking whether the set $\left\{<x,y>: \varphi_x(y)\downarrow\right\}$ is recursive, i.e., whether it can be effectively decided if the function φ_x is defined for an argument y? The answer to this problem is negative. Hence the halting problem is undecidable.

Here are some examples of other undecidable problems from the computation theory:

– is $\varphi_x = 0$?,

[13]Let us add that in this field worked also and received some interesting results Polish logician Józef Pepis. He was active at the Jan Kazimierz University in Lvov. In August 1941 Pepis was killed by Gestapo.

- is $\varphi_x = \varphi_y$?,
- does $y \in dom(\varphi_x)$?
- does $y \in rng(\varphi_x)$?
- is $\varphi_x(x) = 0$?
- does $\varphi_x(x)\downarrow$?
- does $\varphi_x(y) = 0$?

And here are some other examples in the language of Turing machines. The following problems are undecidable:

- does a given Turing machine *M* stop on all inputs?
- does for a given Turing machine *M* and a given input *x* there exist an *y* such that *M*(*x*)=*y* ?
- does the computation of the Turing machine *M* on the input *x* use all the states of *M*?
- does for a given Turing machine *M* and for *x* and *y* hold *M*(*x*)=*y* ?

4. Conclusions

As shown above most mathematical theories are undecidable. This means that for such theories sets of Gödel numbers of their theorems are not recursive, hence not definable (strongly representable) in Peano arithmetic. This indicates some limits in defining notions in formal systems. An interesting comment to this was given by W.O.V. Quine who said that those systems wanted to swallow a greater piece of ontology than they were able to digest. On the other hand, the cardinality argument shows that this phenomenon is quite normal: in fact, there are uncountably many subsets of the set of natural numbers while the set of formulas of the language of Peano arithmetic, hence the set of definable subsets of the set of natural numbers, is countable. What is surprising here is that among sets of natural numbers that are not definable (not representable) in arithmetic are sets of Gödel numbers of theorems of most mathematical theories.

The fact that most mathematical theories are undecidable should not be astonishing. Indeed, problems formulated in decidable theories are not any longer scientific problems – they can be solved (at least theoretically) in a mechanical way. On the other hand, since the decision procedures are usually of an exponential complexity, they are not practically applicable. So one comes to the conclusion that the mind of a mathematician cannot be replaced by a machine (even in the case of a decidable theory). But why does the human mind overcome a machine? Does the reason for that lie in the fact that a human being is able to perform infinite operations? And maybe it does not work algorithmically but in a creative way, it can move to higher levels (to use higher types) and in this way find solutions inaccessible at lower levels?

The final (and in a sense optimistic) conclusion can be that there will always be open problems mathematicians can work on, there will always be a need for creative thinking in mathematics. Tarski put it in the following amusing way (cf. [21], p. 166):

> I have no doubt that many mathematicians experienced a profound feeling of relief when they heard of this result. Perhaps sometimes in their sleepless nights they thought with horror of the moment when some wicked metamathematician would find a positive solution of the problem, and design a machine which would enable us to solve any mathematical problems in a purely mechanical way, so that any further creative mathematical thought would become a worthless hobby. The danger is now over, that such a robot will ever be created; mathematicians have regained their *raison d'être* and can sleep quietly.

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