



## Superquadratically convergent methods for minimization functions

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### Abstract

In the paper locally superquadratically convergent methods for minimization functions are considered. Threefold symmetric approximations to partial derivatives of the third order are constructed.

### 1. Introduction

Let  $f: D \subset \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $f \in C^3(D)$ ,  $D$  - open set. We want to find  $x^* \in D$  such that  $\nabla f(x^*) = 0$ . For a given  $x_0 \in D$  the Newton method defines the sequence  $\{x_k\}$  in the following way

$$\nabla^2 f(x_k) s_k = -\nabla f(x_k), \quad x_{k+1} = x_k + s_k, \quad k = 0, 1, 2, \dots \quad (1)$$

If the matrix  $\nabla^2 f(x^*)$  is nonsingular then Newton method is locally quadratically convergent to  $x^*$ , i.e. there exist  $c > 0$  and  $\varepsilon > 0$  such that, if  $\|x^* - x_0\| < \varepsilon$ , then

$$\|x_{k+1} - x^*\| \leq c \|x_k - x^*\|^2. \quad (2)$$

To assure global convergence of the method one should consider a sequence

$$x_{k+1} = x_k + t_k s_k, \quad t_k \in \mathbb{R}, \quad k = 0, 1, 2, \dots \quad (3)$$

and the parameter  $t_k$  should satisfy the global convergence conditions. If the matrix  $\nabla^2 f(x^*)$  is singular, then the Newton method is divergent or at most linearly convergent to  $x^*$ . To assure a great speed of convergence for singular problems one applies the method of the third rate of convergence: for a given  $x_0 \in D$  the sequence  $\{x_k\}$  is defined as

$$x_{k+1} = x_k + s_k, \quad k = 0, 1, 2, \dots \quad (4)$$

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where  $s_k$  is the solution of the system of quadratic equations

$$\nabla f(x_k) + \nabla^2 f(x_k)s_k + \frac{1}{2}(\nabla^3 f(x_k)s_k, s_k) = 0. \tag{5}$$

When the calculation of the operator  $\nabla^3 f(x_k)$  is too expensive or is not attainable for computation then we propose a new class of the methods which are locally superquadratically convergent to  $x^*$ , i.e.

$$\lim_{k \rightarrow \infty} \frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|^2} = 0. \tag{6}$$

Let  $B_k = (B_k^1, B_k^2, \dots, B_k^n)$ ,  $B_k^i \in R^{n \times n}$ ,  $B_k^i = (B_k^i)^T$ ,  $i = 1, 2, \dots, n$ . The sequence  $\{x_k\}$  is defined by (1.4) and

$$\nabla f(x_k) + \nabla^2 f(x_k)s_k + \frac{1}{2}(B_k s_k, s_k) = 0. \tag{7}$$

If the problem  $\min_{x \in D} f(x)$  is regularly singular at  $x^*$ , i.e.

$$\det(\nabla^2 f(x^*)) = 0, \text{ and } \|\nabla f(x)\| \geq c \|x - x^*\|^2, \quad c > 0, \quad x \in D, \tag{8}$$

then the sequence  $\{x_k\}$  defined by (1.4) and (1.7) is locally superlinearly convergent to  $x^*$ , if the operators  $B_k$  are constructed in an adequate way. In this paper such algorithms are given.

### 2. The BFGS method

The DFP (Davidon [1], Fletcher and Powell [2]) method is very well known as the method of approximation to the Hessian  $\nabla^2 f(x_k)$ . This formula has the form

$$B_{k+1} = B_k - \frac{B_k s_k s_k^T B_k}{s_k^T B_k s_k} + \frac{y_k y_k^T}{y_k^T s_k}, \quad y_k = \nabla f(x_{k+1}) - \nabla f(x_k). \tag{9}$$

The DFP formula, for nonsingular problems, guarantees local superlinear convergence of the method

$$x_{k+1} = x_k + s_k, \quad B_k s_k = -\nabla f(x_k), \quad k = 0, 1, 2, \dots \tag{10}$$

We may use the DFP formula to approximate the operator  $\nabla^3 f(x_k)$ . Namely, let

$$B_k = (B_k^1, B_k^2, \dots, B_k^n), \quad B_k^i \in R^{n \times n}, \quad B_k^i = (B_k^i)^T, \quad i = 1, 2, \dots, n \tag{11}$$

and let  $\nabla_i^2 f(x_k)$  denote  $i$ -th column of the matrix  $\nabla^2 f(x_k)$ ,  $y_k^i = \nabla_i^2 f(x_{k+1}) - \nabla_i^2 f(x_k)$ . Then

$$B_{k+1}^i = B_k^i - \frac{B_k^i s_k s_k^T B_k^i}{s_k^T B_k^i s_k} + \frac{y_k^i (y_k^i)^T}{(y_k^i)^T s_k}, \quad i = 1, 2, \dots, n. \tag{12}$$

Note that the operators  $B_{k+1}$  satisfy the equation

$$B_{k+1}s_k = \nabla^2 f(x_{k+1}) - \nabla^2 f(x_k), \quad k = 0, 1, 2, \dots \tag{13}$$

Now, from the general theory for the systems of nonlinear equations [3], [4] local superquadratic convergence of the method results (4), (7) with the update (12). On the other hand, the DFP updates do not satisfy the following properties of the partial derivatives

$$\frac{\partial^3 f(x)}{\partial x_i \partial x_j \partial x_l} = \frac{\partial^3 f(x)}{\partial x_i \partial x_l \partial x_j} = \dots = \frac{\partial^3 f(x)}{\partial x_l \partial x_j \partial x_i}, \quad i, j, l = 1, 2, \dots, n. \tag{14}$$

In this case, we say the operator  $\nabla^3 f(x)$  is threefold symmetric (T-symmetric).

It is worth remarking that the operator  $\nabla^3 f(x)$  has only  $P(n) = \frac{1}{6}n(n+1)(n+2)$

different elements and the DFP approximations have  $Q(n) = \frac{1}{2}n^2(n+1)$  different elements, which means that the BFGS formula is not adequate for approximation to  $\nabla^3 f(x)$ . In the next Section we give a new formula for the update of  $B_k$  and  $B_k$  will be T-symmetric.

### 3. New approximation to $\nabla^3 f(x)$

The approximation  $B_k$  to  $\nabla^3 f(x)$  satisfies secant equation (13) and operators  $B_k$  should be threefold symmetric. If we take

$$B_{k+1} = B_k + E, \tag{15}$$

then

$$Es_k = \nabla^2 f(x_{k+1}) - \nabla^2 f(x_k) - B_k s_k = Y. \tag{16}$$

In that case we have to solve the problem

$$\min \|E\|^2, \quad \|E\|^2 = \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n E_{ijk}^2 \tag{17}$$

under constraints

$$Es = Y, \quad s \in R^n, \quad Y \in R^{n \times n}, \quad Y = Y^T \tag{18}$$

and

$$E_{ijk} = E_{ikj} = E_{jik} = E_{jki} = E_{kij} = E_{kji}, \quad i, j, k = 1, 2, \dots, n. \tag{19}$$

**Remark.** If we take another norm of the operator  $E$ , then we get another formula for the update  $B_k$ .

Let  $\Lambda \in R^{n \times n}$ . In our case the lagrangian has the form

$$L(E, \Lambda) = \frac{1}{2} \|E\|^2 + \sum_{i=1}^n \sum_{j=1}^n \Lambda_{ij} \left( \sum_{k=1}^n E_{ijk} s_k - Y_{ij} \right). \tag{20}$$

From this we have

$$\frac{\partial L(E, \Lambda)}{\partial E_{pqr}} = E_{pqr} + \Lambda_{pq} s_r = 0. \quad (21)$$

The fact  $E_{pqr} = E_{qpr}$  implies  $\Lambda = \Lambda^T$ . Since the operator  $E$  is threefold symmetric then equation (21) may be written as

$$E_{pqr} = -\frac{1}{3}(\Lambda_{pq} s_r + \Lambda_{pr} s_q + \Lambda_{qr} s_p) \text{ for } 1 \leq p \leq q \leq r \leq n. \quad (22)$$

Now, the equation  $Es = Y$  has the form

$$\sum_{l=1}^n (\Lambda_{ij} s_l + \Lambda_{il} s_j + \Lambda_{jl} s_i) s_l = -3Y_{ij} \quad 1 \leq i \leq j \leq n \quad (23)$$

or is in the matrix form

$$\Lambda \|s\|^2 + \Lambda s s^T + s s^T \Lambda = -3Y. \quad (24)$$

Therefore

$$s^T \Lambda s = -\frac{1}{\|s\|^2} s^T Y s, \quad u = \Lambda s = \frac{1}{2\|s\|^2} (-3Ys + \frac{s^T Y s}{\|s\|^2} s). \quad (25)$$

Finally

$$\Lambda = -\frac{1}{\|s\|^2} (3Y + u s^T + s u^T). \quad (26)$$

To calculate the new threefold symmetric update  $B_{k+1} = B_k + E$  we use the formulae (22), (25) and (26).

#### 4. Remarks on the local superquadratic convergence of the method

At first we describe the proposed algorithm:

a) Let  $x_0 \in R^n$  and  $B_0 = (B_0^1, B_0^2, \dots, B_0^n)$  - threefold operator be given. Let  $k = 0$ ,

b) Solve, using for example the Newton method, the system of quadratic equations

$$\nabla f(x_k) + \nabla^2 f(x_k) s_k + \frac{1}{2} (B_k s_k, s_k) = 0,$$

c) Calculate  $x_{k+1} = x_k + s_k$ ,  $\nabla f(x_{k+1})$ ,  $\nabla^2 f(x_{k+1})$ ,

d) Update the operator  $B_k$  using the formulae from Section 3,

e) If a stop criterion is not satisfied, then  $k := k + 1$  and return to point b.

To explain a character of convergence of the method we introduce some notations. Let

$$H_k = \int_0^1 \nabla^3 f(x_k + t(x_{k+1} - x_k)) dt, \quad (27)$$

$$L_k = \{X \in R^{n \times n \times n} : X S_k = \nabla^2 f(x_{k+1}) - \nabla^2 f(x_k)\}, \quad (28)$$

$$Q = \{X \in R^{n \times n \times n} : X \text{ is threefold symmetric operator}\}. \quad (29)$$

The set  $Q$  is linear subspace in  $R^{n \times n \times n}$  and  $H_k \in Q$ . Applying Theorem 3.2.7 [5] we have

$$H_k S_k = \nabla^2 f(x_{k+1}) - \nabla^2 f(x_k), \quad (30)$$

which means that  $Q \cap L_k$  is a nonempty linear set. The proposed norm is generated by inner product, so the operator  $B_{k+1}$  is defined as the orthogonal projection of the operator  $B_k$  onto the set  $Q \cap L_k$ , and from Pitagoras Theorem (see [3]) we get

$$\|B_{k+1} - H_k\|^2 + \|B_k - B_{k+1}\|^2 = \|B_k - H_k\|^2 \quad k = 0, 1, 2, \dots \quad (31)$$

The inequality  $\|B_{k+1} - H_k\| \leq \|B_k - H_k\|$  implies local linear convergence of the sequence  $\{x_k\}$ . From equations (31) it results additionally

$$\sum_{k=0}^{\infty} \|B_{k+1} - B_k\|^2 < \infty. \quad (32)$$

The last inequality and the secant equation (13) assure local superquadratic convergence of the proposed algorithm [4].

### References

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