Church's thesis and its epistemological status

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Abstract

The aim of this paper is to present the origin of Church's thesis and the main arguments in favour of it as well as arguments against it. Further the general problem of the epistemological status of the thesis will be considered, in particular the problem whether it can be treated as a definition and whether it is provable or has a definite truth-value.

1. Origin and various formulations of Church's thesis

The Church's thesis is one of the widely discussed statements in the recursion theory and computability theory. For the first time formulated in 1936, it still focuses interest of specialists in foundations and philosophy of computer science. It can be stated simply as the equation

$$\mathcal{O} = \mathcal{R}$$

where $\mathcal{O}$ denotes the class of all (effectively) computable functions and $\mathcal{R}$ the class of all recursive functions. So it says that

a function is (effectively) computable if and only if it is recursive.[1]

The central notion we should start with is the notion of an algorithm. By an algorithm we mean an effective and completely specified procedure for solving problems of a given type. Important is here that an algorithm does not require creativity, ingenuity or intuition (only the ability to recognize symbols is assumed) and that its application is prescribed in advance and does not depend upon any empirical or random factors.

A function $f : N^k \to N$ is said to be effectively computable (shortly: computable) if and only if its values can be computed by an algorithm. So

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[1] The exact definition of recursive functions and properties of them can be found in various books devoted to the recursion theory – cf., e.g., [1] or [2].

consequently, a function $f : N^k \to N$ is computable if and only if there exists a mechanical method by which for any k-tuple $a_1,\ldots, a_k$ of arguments the value $f(a_1,\ldots, a_k)$ can be calculated in a finite number of prescribed steps. Three facts should be stressed here:

– no actual human computability or empirically feasible computability is meant here,
– functions are treated extensionally, i.e., a function is identified with an appropriate set of ordered pairs; consequently the following function

$$f(x) = \begin{cases} 1, \text{ if Riemann's hypothesis is true,} \\ 0, \text{ otherwise} \end{cases}$$

is computable since it is either the constant function 0 or the constant function 1 (so a classical notion of a function is assumed and not the intuitionistic one – see below),
– the concept of computability is a modal notion ("there exists a method", "a method is possible").

Many concrete algorithms have long been known in mathematics and logic. In any such case the fact that the alleged procedure is an algorithm is an intuitively obvious fact. A new situation appears when one wants to show that there is no algorithm for a given problem – a precise definition of an algorithm is needed now. This was the case of Gödel's incompleteness theorems and Hilbert's programme of justification of classical mathematics by finitary methods *via* proof theory. Hence in the thirties of the last century some proposals to precize the notion of an effectively computable function appeared – there were the proposals by Alonzo Church [3], Emil Post [4] and Alan Turing [5] (all were published in 1936 (sic!)).

A natural problem of adequacy of those proposals arose. The positive answer to this question is known in the literature as Church's thesis (sometimes called also Church-Turing thesis). It was formulated for the first time by A.Church in [4] and appeared in the context of research done by himself and his students (among them was S.C.Kleene) on the λ-definability. They studied computable functions asking whether they are λ-computable. In §S7 of [4] Church wrote:

We now define the notion, already discussed, of an effectively calculable function of positive integers by identifying it with the notion of a recursive function of positive integers (or of a λ-definable function of positive integers). This definition is thought to be justified by the considerations which follow, so far as positive justification can ever be obtained for the solution of a formal definition to correspond to an intuitive notion.

The very problem of the adequacy appeared in conversation of Church and Gödel – as has been explained by Church in [3] in the footnote where he wrote:

The question of the relationship between effective calculability and recursiveness (which it is proposed to answer by identifying the two

notions) was raised by Gödel in conversation with the author. The corresponding question of the relationship between effective calculability and λ-definablility had previously been proposed by the author independently.

Though published in 1936, the thesis was announced to the mathematical world by Church already on 19th April 1935 at a meeting of the American Mathematical Society in New York City in a ten minute contributed talk. In the abstract received by the Society on 22nd March 1935 (cf. [6]) one can read:

> Following a suggestion of Herbrand, but modifying it in an important respect, Gödel has proposed (in a set of lectures at Princeton, N.J., 1934) a definition of the term *recursive function*, in a very general sense. In this paper a definition of *recursive function of positive integers* which is essentially Gödel's is adopted. And it is maintained that the notion of an effectively calculable function of positive integers should be identified with that of a recursive function, since other plausible definitions of effective calculability turn out to yield notions which are either equivalent to or weaker than recursiveness.

Add that the term "Church's thesis" appeared for the first time in S.C.Kleene's monograph [7] (on page 317) published in 1952 (earlier, in the paper [8] from 1943 he referred to it as "Thesis I").

A thesis similar to Church's thesis was independently formulated by Alan Turing in [5]. He introduced there what is called today Turing machines and stated that:

> The computable numbers include all numbers which could naturally be regarded as computable.

Explain that "computable numbers" are those reals whose decimal extension is computable by a Turing machine.

Later other formulations similar (and equivalent) to Church's thesis as well as its some paraphrases were proposed. Let us mention here the formulation of Kleene (cf. [8]) where one finds Thesis I stating that:

> Every effectively calculable function (effectively decidable predicate) is general recursive.

A.A.Markov in the book [9] formulated a principle equivalent to Church's thesis. It is stated in the language of algorithms (in the precise sense of Markov!) and says that every algorithm in the alphabet *A* is fully equivalent with respect to *A* to a normal algorithm over *A*.

As examples of some paraphrases of the considered thesis let us quote Boolos and Jeffrey' [10] where they write:

> […] the set of functions computable in our sense [i.e., by a Turing machine – R.M.] is identical with the set of functions that men or machines would be able to compute by whatever effective method, if limitations on time, speed, and material were overcome [p. 20]

and

> […] any mechanical routine for symbol manipulation can be carried out in
> effect by some Turing machine or another. [p. 52]

Some other paraphrases can be found in L.Kalmár [11], G.Kreisel [12] or in
D.R.Hofstadter [13].

2. Attempts to justify Church's thesis

The natural question that should be asked is the following: is Church's thesis
true (correct, sound)?

Let us start by noticing that the thesis is widely accepted today. Within
recursion theory a technique (called sometimes "argument by Church's thesis")
has been developed. It consists in concluding that a function is recursive, or that
a set is decidable, just because there is an algorithm for it. Examples of its
applications can be found in the monograph by H.Rogers [14].

Note also that Church's thesis has a bit different meaning when considered
from the constructivist (or intuitionistic) position. Indeed, for a constructivist a
formula $\forall x \exists ! \Phi(x, y)$ (hence a definition of a function) states that given an x
one *can find* a unique y such that $\Phi(x, y)$. This amounts to an assertion that a
certain function is computable. Thus Church's thesis is the statement that if
$\forall x \exists ! \Phi(x, y)$ then there is a *recursive function f* such that $\forall x \Phi(x, f(x))$. Hence
for a constructivist Church's thesis is a claim that *all* number-theoretic functions
are recursive.

Church's thesis is a statement formulated rather in a colloquial language than
in the precise language of mathematics – it is about the class of computable
functions defined in the colloquial, pragmatic, pre-theoretic language using the
imprecise notion of an algorithm or an effective mechanical method. Hence it
seems that *a priori* no precise mathematical proof of this thesis can be expected.
The only thing one can do is to try to indicate evidences confirming it (or
arguments against it). All such arguments must, at least in part, be of a
philosophical character – they cannot be purely mathematical arguments.

Since one cannot give a precise mathematical proof of the thesis of Church,
let us look for arguments in favour of it (arguments against it will be indicated in
the sequel). Further, since the inclusion $\mathcal{R} \subseteq \mathcal{O}$ is obvious, the essential part of
Church's thesis is the inclusion $\mathcal{O} \subseteq \mathcal{R}$. The arguments in favour of it can be
divided into three groups:

(A) heuristic arguments (no counterexamples arguments),
(B) direct arguments,
(C) arguments based on the existence of various specifications of the notion
     of computability.

Among arguments of group (A) one can find the following ones:

(A$_1$) all particular computable functions occurring in mathematics were shown to be recursive,

(A$_1'$) no example of a computable function not being recursive was given,

(A$_2$) it was shown that all particular methods of obtaining computable functions from given computable ones lead also from recursive functions to recursive functions,

(A$_2'$) no example of a method leading from computable functions to computable functions but not from recursive functions to recursive ones was given.

One can easily see that the above arguments are in a certain sense of empirical (or quasi-empirical) character.

Arguments of group (B) consist of theoretical analyses of the process of computation and attempts to show in this way that only recursive functions can be computable. Such was the argumentation of Turing in [5] where a detailed analysis of the process of computing a value of a function was given and where the conclusion was formulated that any possible computation procedure has a faithful analogue in a Turing machine and that, therefore, every computable function is recursive.

Arguments of group (C) are based on the fact that in the second quarter of this century several somewhat different mathematical formulations of computability were given (more or less independently). All of them have been proven to be extensionally equivalent and equal to the class $\mathcal{R}$ of recursive functions. It suggests that their authors have the same intuitions connected with the notion of computability. The equality of all those classes of functions can serve as an argument in favour of the thesis that the class $\mathcal{R}$ comprises all computable functions.

All attempts to define the notion of computability in the language of mathematics can be classified in the following way:

1. algebraical definitions – they consist of fixing certain initial functions and certain operations on functions. One considers then the smallest class of functions containing the initial functions and closed under the indicated operations;

2. definitions using abstract mathematical machines – as an example of such a definition there can serve the definition of the class of functions computable in the sense of Turing (cf. [5]);

3. definitions using certain formal systems – an example of a definition of this sort is the definition of functions computable in the sense of Markov (i.e., of functions computable by normal Markov's algorithms, cf. [9]), Herbrand-Gödel-Kleene definition of computable functions (cf. [15] and

Kleene [7]), definition of computability by representability in a formal system, the theory of λ-definability by Church (cf. Church [16]) or the theory of Post normal systems (cf. Post [17]).

Add also the following pragmatic argument due to M.Davis who writes in [18]:

The great success of modern computers as all-purpose algorithm-executing engines embodying Turing's universal computer in physical form, makes it extremely plausible that the abstract theory of computability gives the correct answer to the question "What is a computation?" and, by itself, makes the existence of any more general form of computation extremely doubtful.

Having considered arguments in favour of Church's thesis let us study now arguments against it. One can find various such arguments but, as Shapiro writes in [19] (p. 354), "[they] seem to be supported only by their authors". We shall discuss here only arguments of Kalmár formulated by him in [11].

Kalmár gives of course no example of a particular function which is not recursive but for which there exists a mechanical method of calculating its values. He shows only that Church's thesis implies certain peculiar consequences.

Let $F(k,x,y)$ be a ternary partially recursive function universal for binary recursive functions. Let $f(x,y)$ be its diagonalization, i.e., a function defined as:

$$f(x,y) = F(x,x,y).$$

The function f is recursive. Consider now the following example of a nonrecursive function (given by Kleene – cf. [20], Theorem XIV).

$$g(x) = \mu y\big(f(x,y) = 0\big) = \begin{cases} \text{the smallest } y \text{ such that} \\ f(x,y) = 0, \text{ if such } y \text{ exists,} \\ 0, \text{ otherwise.} \end{cases}$$

The function g is nonrecursive, hence by Church's thesis it is also noncomputable. On the other hand, for any natural number p for which there exists a number y such that $f(p,y) = 0$, there is a method of calculating the value of $g(p)$, i.e., of computing the smallest number y such that $f(p,y) = 0$. Indeed, it suffices to compute successively $f(p,0)$, $f(p,1)$, $f(p,2)$, … (each of them can be calculated in a finite number of steps since f is recursive) till we get the value 0. For any number p for which it can be proved (by any correct method) that there exists no y such that $f(p,y) = 0$, we also have a method of calculating the value of $g(p)$ in a finite number of steps – it suffices simply to prove that there exists no number y such that $f(p,y) = 0$ which requires a finite number of steps and leads to the result $g(p) = 0$. Hence assuming that the function g is not computable and using *tertium non datur* (note that it was already used in the definition of g/) one comes to the conclusion that there are natural numbers p such that, on the one

hand, there is no number *y* with the property *f(p,y)* = 0 and, on the other hand, this fact cannot be proved by any correct means. So Church's thesis implies the existence of absolutely undecidable propositions which can be decided! An example of such a proposition is any sentence of the form $\exists y \left[f(p,y) = 0 \right]$ where *p* is a number for which there is no *y* such that *f(p,y)* = 0. Hence it is an absolutely undecidable proposition which we can decide, for we know that it is false!

In the above considerations we used a somewhat imprecise notion of provability by any correct means. This imprecision can be removed by introducing a particular formal system (more exactly: a system of equations) and showing that the considered sentence cannot be proved in any consistent extension of this system.

But the situation is even more peculiar. Church's thesis not only implies that the existence of false sentences of the form

$$\exists y \left[f(p,y) = 0 \right]$$

is absolutely undecidable, but also that the absolute undecidability of those sentences cannot be proved by any correct means. Indeed, if *P* is a relation and the sentence $\exists y P$ is true then there exists a number *q* such that *P(q)*. So the question: "Does there exist a *y* such that *P(y)*?" is decidable and the answer is YES (because $P(q) \rightarrow \exists y P(y)$. Hence if the sentence $\exists y P(y)$ is undecidable then it is false. So if one proved the undecidability of the sentence $\exists y P(y)$ then one could also prove that $\neg \exists y P(y)$. Hence one would decide the undecidable sentence $\exists y P(y)$, which is a contradiction. Consequently, the undecidability of the sentence $\exists y P(y)$ cannot be proved by any correct means.

Kalmár comes to the following conclusion: "there are pre-mathematical notions which must remain pre-mathematical ones, for they cannot permit any restriction imposed by an exact mathematical definition" (cf. [11], p. 79). Notions of effective computability, of solvability, of provability by any correct means can serve here as examples.

Observe that one can treat the argumentation of Kalmár given above not as an argumentation against Church's thesis but as an argumentation against the law of excluded middle (*tertium non datur*) – this law played a crucial role in Kalmár's argumentation. So did, for example, Markov.

Arguments against Church's thesis were formulated also by R.Péter [21], J.Porte [22] and G.L.Bowie [23]. The latter claims that the notion of an effective computability is intensional whereas the concept of recursiveness is extensional. Hence they cannot be identical and consequently Church's thesis is possibly false. Note (in favour of the intensionality of the concept of computability) that the existence of a computation of (a value of) a function depends not only on the

description of the function but also on the admissible notation for inputs and outputs.

3. Epistemological status of Church's thesis

Let us look now at Church's thesis from the epistemological point of view and ask: what does this thesis really mean, is it true or false or maybe has no truth-value at all, is this problem decidable, and if yes, then where and by which means can it be decided?

Start by noting that computability is a pragmatic, pre-theoretical concept. It refers to human (possibly idealized) abilities. Hence Church's thesis is connected with philosophical questions about relations between mathematics and material or psychic reality. There are scholars who treat the notion of computability as a notion of a psychological nature. For example E. Post wrote [24], p. 408 and 419):

> […] for full generality a complete analysis would have to be made of all the possible ways in which the human mind could set up finite processes for generating sequences. […] we have to do with a certain activity of the human mind as situated in the universe. As activity, this logico-mathematical process has certain temporal properties; as situated in the universe it has certain spatial properties.

On the other hand, the concept of computability has modal character whereas the notion of a recursive function (or of a function computable by Turing machine) is not a modal one. Hence Church's thesis identifies the extension of an idealized, pragmatic and modal property of functions with the extension of a formal, precisely defined *prima facie* non-modal arithmetical property of functions. Consequently Church's thesis is a proposal to exchange modality for ontology.

In a philosophical tradition one has two approaches to this problem: (1) One of them, traced to W.V.O. Quine, is skeptical of modal notions altogether and suggests that they are too vague or indeterminate for respectable scientific (or quasi-scientific) use. On such a view Church's thesis has no definite truth-value, so it is neither true nor false. (2) Other tradition, while not so skeptical of modality as such, doubts that there can be any useful reduction of a modal notion to a non-modal one.

S. Shapiro (cf. [25]) claims that both those traditions should be rejected while considering Church's thesis. In this case the problem of modality is solved by assuming that Turing machines represent in a certain sense all possible algorithms or all possible machine programmes and sequences of Turing machine configurations represent possible computations. Hence Church's thesis would hold only if, for every possible algorithm there is a Turing machine that represents an algorithm that computes the same function. So we have a thesis that the possibilities of computation are reflected accurately in a certain

arithmetic or set-theoretic structure. Similarly for λ-definability and for recursive functions. This leads us to the so called Church's superthesis (formulated for the first time by G. Kreisel in [26] (p.177) in the following way:

> […] the evidence for Church's thesis, which refers to *results*, to functions computed, actually establishes more, a kind of *superthesis*: to each […] algorithm […] is assigned a […] [Turing machine] programme, modulo trivial conversions, which can be seen to define the same computation process as the [algorithm].

Pragmatic modal notions do not have sharp boundaries. There are usually bordeline cases. Similar situation is of course also in the case of Church's thesis – recall the idealized character of an unequivocal notion of computability. In general vague properties cannot exactly coincide with a precise one. Since computability is a vague notion and recursiveness (and other equivalent notions) is a precise one, hence Church's thesis does not literally have a determinate truth-value or else it is false. Church's thesis might then be treated as a proposal that recursiveness be substituted for computability for certain purposes and in certain contexts.

Church's thesis is suited for establishing negative results about computability. When one shows that a given function is not recursive, then one can conclude that it cannot be computed. On the other hand, if the function has been shown to be recursive, then this gives no information on the feasibility of an algorithm and does not establish that the function can be calculated in any realistic sense.

Church's thesis can also be treated simply as a definition. If we treat it as a nominal definition then its acceptance (or rejection) is a matter of taste, convenience, etc. If accepted, it is vacuously true, if rejected, there is no substantive issue. Add however that such an approach to Church's thesis cannot be found in the literature – it is a subject of philosophical and mathematical studies what proves that it is not treated as a nominal definition only.

Note that Church proposing for the first time the thesis thought of it just as a definition. In fact in the paper [3] from 1936 he wrote:

> The purpose of the present paper is to propose a definition of effective calculability […].

Recall also his words from §7 of [3] quoted above where he said:

> *We now define* [emphasis is mine – R.M.] the notion, already discussed, of an effectively calculable function of positive integers by identifying it with the notion of a recursive function […]

But it should be stressed that Church understood here the definition as a real one (and not as a nominal definition), i.e., as an explication or rational reconstruction.

Turing in [5] – though he did not use the term "definition", but writing that he wants to show "that all computable numbers are [Turing] 'computable'" – clearly regarded it just as the definition of computability.

Also by Gödel one finds words which may suggest that he treated Church's thesis as a definition. In fact in [27] he wrote:

> […] one has for the first time succeeded in giving an absolute definition of an interesting epistemological notion, i.e. one not depending on the formalism chosen.

And in [28] he wrote:

> The greatest improvement was made possible through the precise definition of the concept of finite procedure […] This concept, […] is equivalent to the concept of a "computable function of integers" […] The most satisfactory way, in my opinion, is that of reducing the concept of finite procedure to that of a machine with a finite number of parts, as has been done by the British mathematician Turing.

E.Post treated Church's thesis as "a working hypothesis" and as "a fundamental discovery in the limitations of the mathematicizing power of Homo Sapiens". He was convinced that in the process of confirming it can receive the status of "a natural law" (cf. [4]). Add that he warned against treating Church's thesis as a defiinion because it would dispense us from the duty and necessity of looking for its confirmation.

The standard approach to Church's thesis hold it to be a rational reconstruction (in the sense of R.Carnap and C.Hempel). A rational reconstruction is a precise, scientific concept that is offered as an equivalent of a prescientific, intuitive, imprecise notion. It is required here that in all cases in which the intuitive notion is definitely known to apply or not to apply, the rational reconstruction should yield the same outcome. In cases where the original notion is not determinate, the reconstruction may decide arbitrarily. Confirmation of the correctness of a rational reconstruction must involve, at least in part, an empirical investigation. And, what is more important, it cannot be proved. So was for example the opinion of Kleene who wrote in [7] (pp. 317-319):

> Since our original notion of effective calculability of a function is a somewhat vague intuitive one, [Church's thesis] cannot be proved […] While we cannot prove [Church's thesis], since its role is to delimit precisely a hirthto vaguely conceived totality, we require evidence that it cannot conflict with the intuitive notion which it is supposed to complete; i.e. we require evidence that every particular function which our intuitive notion would authenticate as effectively calculable is [recursive].

Similar was the opinion of L.Kalmár who wrote in [11]:

> Church's thesis is not a mathematical theorem which can be proved or disproved in the exact mathematical sense, for it states the identity of two notions only one of which is mathematically defined while the other is used by mathematicians without exact definition.

J.Folina claims in [29] that Church's thesis is true (since "there is a good deal of convincing evidence" – cf. [29], p.321) but that it is not and cannot be mathematically proved.

Talking about "mathematical provability" one should beforehand explain clearly what it exactly means. This is stressed by S.Shapiro in [25]. He says that there exists of course no ZFC proof of Church's thesis, there is also no formal proof of it in a deductive system.

To prove (in a formal way) Church's thesis one should construct a formal system in which the concept of computability would be among primitive notions and which would be based (among others) on axioms characterizing this notion. The task would be then to show that computability characterized in such a way coincides with recursiveness. But another problem would appear now: namely the problem of showing that the adopted axioms for computability do, in fact, reflect exactly properties of (intuitively understood) computability. Hence we would arrive at Church's thesis again, though at another level and in another context.

On the other hand it should be stressed that in mathematics one has not only formal proofs, there are also other methods of justification accepted by mathematicians. Taking this into account E.Mendelson comes in [30] to the conclusion that it is completely unwarrented to say that "CT [i.e., Church's thesis – R.M.] is unprovable just because it states an equivalence between a vague, imprecise notion (effectively computable function) and a precise mathematical notion (partial-recursive function)". And adds:

> My viewpoint can be brought out clearly by arguing that CT is another in a long list of well-accepted mathematical and logical "theses" and that CT may be just as deserving of acceptance as those theses. Of course, these theses are not ordinarily called "theses", and that is just my point.

As a justification of his claim Mendelson considers several episodes from the history of mathematics, in particular the concept of function, of truth, of logical validity and of limits. In fact, till the 19th century a function was tied to a rule for calculating it, generally by means of a formula. In the 19th and 20th centuries mathematicians started to define a function as a set of ordered pairs satisfying appropriate conditions. The identification of those notions, i.e., of an intuitive notion and the precise set-theoretical one, can be called "Peano thesis". Similarly "Tarki's thesis" is the thesis identifying the intuitive notion of truth and the precise notion of truth given by Tarski. The intuitive notion of a limit widely used in mathematical analysis in the 18th century and then in the 19th century applied by A.Cauchy to define basic notions of the calculus has been given a precise form only by K.Weierstrass in the language of ε-δ. There are many other such examples: the notion of a measure as an explication of area and volume, the definition of dimension in topology, the definition of velocity as a derivative, etc. Mendelson argues that, in fact, the concepts and assumptions that support

the notion of recursive function are no less vague and imprecise than the notion of effective computability. They are just more familiar and part of a respectable theory with connections to other parts of logic and mathematics (and similarly for other theses quoted above). Furthermore, the claim that a proof connecting intuitive and precise notions is impossible is false. Observe that the half of Church's thesis, i.e., the inclusion $\mathcal{R} \subseteq \mathcal{O}$ is usually treated as obvious. Arguments are here similar to others used in mathematics (and one uses here two notions only one of which is precisely given). In mathematics and logic proof is not the only way in which a statement comes to be accepted as true. Very often – and so was the case in the quoted examples – equivalences between intuitive notions and precise ones were simply "seen" to be true without proof or based on arguments which were mixtures of intuitive perceptions and standard logical and mathematical reasonings.

Note at the end that there are also theorists who regard Church's thesis as proved. So, for example, R.Gandy says in [31] that Turing's direct argument that every algorithm can be simulated on a Turing machine proves a theorem. He regards this analysis to be as convincing as typical mathematical work.

4. Conclusions

The above survey of opinions on Church's thesis shows that there is no common agreement about its epistemological status. The crucial point are always the philosophical presuppositions concerning, for example, the nature of mathematics and of a mathematical proof. The situation can be summarized as follows: there are some arguments and evidences in favour of Church's thesis (arguments against it are weaker), hence there are reasons to believe it is true. On the other hand, there is no (and there cannot exist any) formal proof (on the basis of, say, Zermelo-Fraenkel set theory or any other commonly accepted basic theory) of the thesis that would convince every mathematician and close the discussions. But this happens not so rarely in mathematics (compare Mendelson's examples given above). Mathematics (and logic) is not only a formal theory but (the working mathematics) is something more. Hence one should distinguish several levels in mathematics (pre-theoretical level, level of formal reconstructions, etc.) and consequently, take into account various ways of justification adopted in them.

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