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Quantum gates and projection evolution

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Abstract

A new model of quantum objects time evolution called the projection evolution is analyzed against a possibility of constructing the basic quantumgates following the Nielsen's scheme.

1. Introduction

One of the main goals of quantum formalism in respect to the basic theory of quantum computers is an analysis of possible quantum gates. There is extensive literature this problem. Short overview and many references one can find e.g. in [1].

The idea of quantum algorithms which can be applied to real quantum computers is directly related to a process of quantum evolution. However, it is very well known in quantum theory that there are two completely different kinds of evolutions. The former can be understood as sequences of unitary transformations generated by the Hamiltonian of the system under consideration and the latter is described by the so called projection postulate [2]. It has been shown that all possible logical operations can be combined as a sequence of e.g. elementary single qubit unitary operations called the Hadamard gate (H) and the phase shift gate (f), and the two-qubit operation known as controlled-NOT (*CNOT*) [3].

In principle, it is possible to construct a complete system consisting of fewer than three types of elementary quantum gates but we are not interested in constructions of minimal systems.

It is important to notice that the quantum gates providing required transformations should be not abstract mathematical ideas but evolving in time quantum devices obeyed by quantum rules. This statement implies a requirement

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of solving some contradictions between the unitary evolution concept and the projection postulate.

The unitary process of evolution, even in more realistic theoretical considerations, seems to be an idealization due to a decoherence phenomenon. A decoherence of quantum states is a very fast process which can destroy the possibility to perform even elementary quantum calculations. In addition, the quantum gates are measurement devices inside which there are performed rather complex, time consuming quantum processes. However, such calculations in very simple cases were made experimentally [4].

We think that these experimental facts and some paradoxes concerning the Schrödinger equation and the projection postulate should open again a discussion about the time evolution of quantum objects.

In the paper [5] we have proposed a new unified evolution law based on the projection postulate including as a special case an unitary type of evolution where the Schrödinger equation is a result of continuous series of projections. The main difference between the standard approach and the projection evolution law is that in our proposal just before the unitary evolution one projection must be performed. It is an important remark because projection requires forces to forget about the previous state of the system and is an irreversible operation. But the quantum gates which are building blocks of quantum computers should be pure unitary devices.

One of the important problems we left open in that paper was a possibility of evolution which leads to pure unitary transformation which, in turn, should allow for construction of quantum logic.

In the present paper we show a simple example of construction of single qubit quantum gates without using unitary form of evolution. We follow the idea by Nielsen [6] who proposed to apply the projection postulate to construct the quantum gates. His algorithm was applied e.g., in [7] to construct a series of quantum gates.

An important feature of the considered algorithm is requirement for existing a decision "device" which after the appropriate measurement can make decision about further procedure.

In the case of the projection evolution, one needs to construct only the appropriate set of projection operators which can be represented by some "measurement type devices" without any "observer" which can decide about further steps of the procedure.

2. Projection evolution

Because the idea seems to be new we sketch the main points of the hypothesis referring to more detailed description for the paper [5].

In general, the states of quantum system can be described by the so called quantum density operators r. In quantum algorithms they are this “medium” which is transformed to get a computational outcome.

In the following by t we will denote an evolution parameter ordering causally related physical events or in the case of quantum computers subsequent steps of computations. In principle, it is enough to consider t to be a real c-number parameter. In our case, however, we do not need to consider t to be a continuous variable and we restrict ourselves to only discrete values of this time-like parameter $t_0 < t_1 < t_2 < \ldots t_n \ldots$.

In addition, for each t we define a family of projections which are orthogonal resolutions of unity i.e., roughly speaking, for each t they fulfil the following conditions:

$$\begin{aligned} &M(t;n)\,M(t;n') = d_{n,n'}\,M(t;n), \\ &\sum_n M(t;n) = \mathbf{I}. \end{aligned} \tag{1}$$

The operators $M(t;n)$ should represent the essential properties of the physical system under consideration responsible for its time evolution. In this sense they play role of some evolution operators.

In place of usual unitary evolution we postulate that for each value of the evolution parameter t the state of the physical system is generated by randomly chosen projection of previous state with one of the operators $M(t;n)$ according to the following probability distribution with respect to choice of the quantum numbers n:

$$\mathrm{Prob}(t_{n+1};n) = \mathrm{Tr}\left[M(t_{n+1};n)\, r(t_n) \right], \tag{2}$$

where $r(t_n)$ denotes the previous state. It roughly means, that we apply the “projection postulate” to get a new state at the evolution parameter t_{n+1}.

The resulting state is of the following form:

$$r(t_{n+1}) = \frac{M(t_{n+1};n_{n+1})\, r(t_n)\, M(t_{n+1};n_{n+1})}{\mathrm{Tr}\left[M(t_{n+1};n_{n+1})\, r(t_n)\, M(t_{n+1};n_{n+1}) \right]}. \tag{3}$$

Equation (3) implies that the state of a physical system for the next instant t_{n+1} is chosen randomly with the probability distribution (2), from all possible states

$$r' = \mathrm{Tr}\left[M(t_{n+1};n_{n+1})\, r(t_n)\, M(t_{n+1};n_{n+1}) \right]^{-1} M(t_{n+1};n_{n+1})\, r(t_n)\, M(t_{n+1};n_{n+1}),$$

where n runs over the whole range required by (1).

The probability of finding a given path of evolution (quantum calculations) at the instant t_n can be easily calculated using equations (3) and (2) by multiplying the appropriate probabilities of choosing subsequent states on the path. This gives the formula:

$$\mathrm{Prob}(t_n;n_1,n_2,\ldots,n_n)=\mathrm{Tr}\Big[\mathcal{M}(t_n;n_n)\mathcal{M}(t_{n-1};n_{n-1})\ldots\mathcal{M}(t_1;n_1)\mathcal{M}(t_0;n_0)\,r_0\\ \mathcal{M}(t_0;n_0)\mathcal{M}(t_1;n_1)\ldots\mathcal{M}(t_{n-1};n_{n-1})\mathcal{M}(t_n;n_n)\Big]. \tag{4}$$

In this way, we have defined the time evolution (quantum computations) as a kind of permanent measurements made by the Nature. Within this idea we treat it as a fundamental process – a new Law of the Nature.

It is important to note that according to this hypothesis quantum computations should be understood as series of stochastic processes. This property should have influence on possibility of building some quantum algorithms, especially their efficiency. In addition, the decoherence process becomes an internal property of the calculations because the decoherence phenomenon is of projection nature.

This way, we are able to describe, in an unified manner, the whole process of quantum calculations.

3. Unitary quantum gates

A similar problem of constructing the quantum gates using only the projection postulate was originally studied by Nielsen [6]. Short description of the idea of Nielsen's scheme one can find in [7]. Following the last paper we briefly review the protocol for the single qubit unitary gate *U*. More precisely having a single qubit state $|y\rangle$ we are interested in construction of the transformed state $U|y\rangle$.

For this purpose first we need to produce off line two ancilla qubits in one of the four orthonormal states $j=(0,1,2,3)$:

$$|U_j\rangle \equiv (I\otimes Us_j)|\mathrm{EPR}\rangle = (I\otimes U)|B_j\rangle, \tag{5}$$

where the $|\boldsymbol{EPR}\rangle$ state can be written in the standard $\{|0\rangle,|1\rangle\}$ as

$$|\boldsymbol{EPR}\rangle \equiv \frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle+|1\rangle|1\rangle\right), \tag{6}$$

and the Bell basis is denoted by

$$|B_i\rangle \equiv (\mathbf{I}\otimes s_i)|\boldsymbol{EPR}\rangle. \tag{7}$$

In equation (7) **I** is the identity operator and s_i where $i=0,1,2,3$ denotes for $i=0$ the unit matrix and for $i=1,2,3$ the standard Pauli matrices, respectively.

Next we perform the Bell measurement on the compound system of the first ancilla and our input state $|y\rangle$. After the measurement, as a result, with probability 1/4, we get one of the four states $|B_m\rangle$ for both measured particles and the second ancilla occurs in the state $Us_j s_m|y\rangle$. For $m=j$ because of standard properties of the Pauli matrices, with probability 1/4, we get the required result, namely, the second ancilla is then in the state $|Uy\rangle$ (eventually up to the phase factor). However, for $m \neq j$ one needs to repeat the procedure using as the input state the vector $U s_j s_m |y\rangle$ trying now to simulate the gate

$$U' = U s_m s_j U^{\dagger}. \tag{8}$$

In average, after four attempts we should be successful in obtaining the output as the input state transformed by the unitary gate *U*.

This scheme allows nearly immediately to construct the appropriate projection evolution operators $\mathcal{M}(t;n)$.

First, we need to prepare from the three particles two ancilla qubits leaving unchanged the state of the third particle we would like to transform. The appropriate decomposition of unity can be written as

$$\mathcal{M}(t=0;n) = \begin{cases} |U_j\rangle_{1,2\ 1,2}\langle U_j| \otimes \mathbf{I}_3, & n=0, \\ \text{STOP}, & n=1, \end{cases} \tag{9}$$

where the label *j* is fixed and equals 0,1,2 or 3. The indices in the vector $|U_j\rangle_{1,2}$ and in the unit operator $\mathbf{I}_3$ denote the labels of the particles on which the projection operators act. The projection operator STOP represents the other possibilities of transformations which effectively stop the particle at the device (gate). Obviously the sum of both projection operators gives the unit operator to have the total probability of all processes equal one.

The next step evolution corresponds to the Bell measurement on the first and third particles:

$$\mathcal{M}(t=1;n) = \begin{cases} |B_m\rangle_{1,3\ 1,3}\langle B_m| \otimes \mathbf{I}_2, & n=m=0,\ldots,3, \\ \text{STOP}, & n=4, \end{cases} \tag{10}$$

In this and other cases the operator STOP is, in general, different for different t.

The last operation of the gate *U* is to filter the unwilling cases giving the required result for the outcome i.e.,\ the output particle should be in the state $U|y\rangle$:

$$\mathcal{M}(t=2;n) = \begin{cases} |B_j\rangle_{1,3\ 1,3}\langle B_j| \otimes \mathbf{I}_2, & n=0, \\ \text{STOP}, & n=1, \end{cases} \tag{11}$$

where *j* is the initial value of label for the first and second particles.

The resulting projection evolution operator $\mathcal{M}(t;n)$, where $t=0,1,2$ and for each t the label n has the values defined above, represents the single qubit unitary gate. The three values of t represent here three intervals of time in which the appropriate processes take place. However, within the above protocol the explicit values of these intervals are irrelevant.

The construction gives rather inefficient device because the probability of getting the required result is not larger than 1/4. Probably there are possibilities to construct more efficient gates, but it was not the purpose of our investigations.

In addition, the longer calculations, using this type of logic quantum gates, require higher intensity of input particles because a part of them is absorbed during the evolution. On this level of analysis it is difficult to say if this is a general property which we have to take into account building quantum computers or not.

The problem require furthers investigations.

References

[1] Keyl M., Phys. Rep. 369, (2002), 431. *See also references in* Cabello A.; http://xxx.lanl.gov arXiv:quant-ph/0012089.
[2] Busch P., Lahti J., Mittelstaedt P., *The Quantum Theory of Measurement*, Lectures notes in physics/M, Springer, (1966).
[3] Berenco A. et al., Phys, Rev. A52 (1995) 3457.
Berenco A., Deutch D., Ekert A., Jozsa R., Phys. Rev. Lett. 74 (1995) 4083.
Lloyd S., Phys. Rev. Lett. 75 (1995) 346.
[4] http://www.ibm.com/news/2000/08/15.phtml
http://www.research.ibm.com/resources/news/20011219_quantum.shtml
[5] Góźdź A., Pietrow M., *Projection evolution in quantum mechanics*; http://xxx.lanl.gov arXiv:quant-ph/0303084, submitted for publication.
[6] Nielsen M.A., http://xxx.lanl.gov arXiv:quant-ph/0108020.
[7] Fenner S.A., Zhang Y., http://xxx.lanl.gov arXiv:quant-ph/0111077.