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On Poncelet's porism

ABSTRACT. We consider circular annuli with Poncelet's porism property. We prove two identities which imply Chapple's, Steiner's and other formulas. All porisms can be expressed in the form in which elliptic functions are not used.

1. Introduction. An annulus formed by two circles will be called circular annulus. Poncelet [1] proved that if there exists one circuminscribed (simultaneously inscribed in the outer and circumscribed on the inner) n-gon in a fixed circular annulus, then any point of the outer circle is the vertex of some circuminscribed n-gon. Circuminscribed polygon in a circular annuli will be called a bicentric polygon. There is a general analytic expression relating the circumradius ρ, inradius r, and offset between the circumcenter and incenter a for a bicentric polygon. The condition for an n-gon to be bicentric is the following

$$sc\left(\frac{1}{n}K\left(k\right),k\right)=\frac{c\sqrt{b^2-d^2}+b\sqrt{c^2-d^2}}{d\left(b+c\right)}, \tag{1.1}$$

where

$$d=\frac{1}{\rho+a},\quad b=\frac{1}{\rho-a},\quad c=\frac{1}{r},$$
$$\omega=\cosh^{-1}\left[1+\frac{2c^2\left(d^2-b^2\right)}{d^2\left(b^2-c^2\right)}\right],\quad k^2=1-e^{-2\omega},$$

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$sc(x,k)$ is a Jacobi elliptic function and $K(k)$ is a complete elliptic integral of the first kind, see [3], [1].

Kerawala [3] established many porisms in a simple explicit form without resorting to the use of elliptic functions. In this paper we will show that it can be done in any case.

The paper [1] very precisely explains all details connected with Poncelet's porism property.

2. Main theorem. Let $S((u_1,u_2),r)$ denote a circle with the center at (u_1,u_2) and the radius r. We consider an annulus $\mathcal{A}$ formed by two circles $S((0,0),r)$ and $S((a,0),\rho)$ where $a>0$ and $a+r<\rho$. Let

$$z(t)=re^{it} \quad \text{for } t\in[0,2\pi] \tag{2.1}$$

be a parametrization of $S((0,0),r)$. The circle $S((d,0),\rho)$ will be considered in the form

$$w(t)=re^{it}+\lambda(t)\,ie^{it} \quad \text{for } t\in[0,2\pi]. \tag{2.2}$$

It is easy to verify that the positive-valued function λ is given by the formula

$$\lambda(t)=\sqrt{a^2\sin^2 t+2ar\cos t+\rho^2-r^2-a^2}-a\sin t \quad \text{for } t\in[0,2\pi]. \tag{2.3}$$

A fixed point $w(t)$ of $S((a,0),\rho)$ belongs to some α-isoptic C_α.

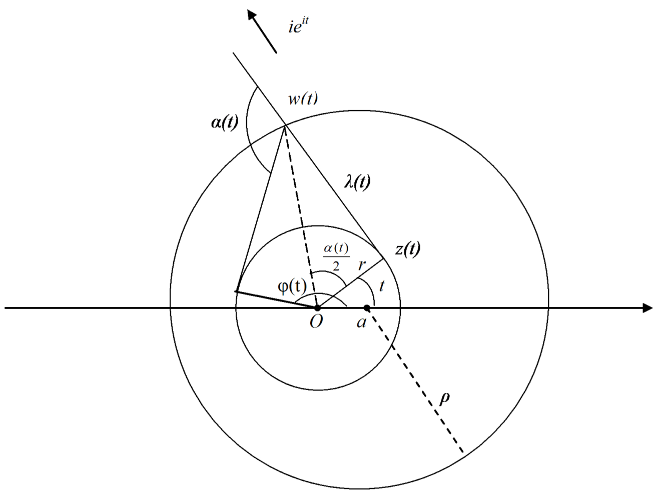

FIGURE 1.

We have

$$\tan\frac{\alpha(t)}{2}=\frac{\lambda(t)}{r}. \tag{2.5}$$

Let

$$\varphi(t)=t+\alpha(t), \tag{2.6}$$

and $\varphi^{[0]}(t) = t$, $\varphi^{[1]} = \varphi$, $\varphi^{[n]} = \varphi\left(\varphi^{[n-1]}\right)$ for $n = 2, 3, \dots$ (see [2, (4.4)]).

The formula (2.5) implies that

$$e^{i\alpha(t)} = \frac{r^2 - \lambda^2(t)}{r^2 + \lambda^2(t)} + i\frac{2\lambda(t)\, r}{r^2 + \lambda^2(t)},$$

i.e.,

$$e^{i\varphi(t)} = \left(\frac{r + \lambda(t)\, i}{|r + \lambda(t)\, i|}\right)^2 e^{it}. \tag{2.7}$$

We consider, for each fixed $t \in [0, 2\pi]$, the following sequence of complex numbers

$$\omega_m(t) = \frac{r + \lambda\left(\varphi^{[m]}(t)\right) i}{\left|r + \lambda\left(\varphi^{[m]}(t)\right) i\right|} \quad \text{for} \quad m = 0, 1, 2, \dots . \tag{2.8}$$

The formula (2.7) can be rewritten in the form

$$e^{i\varphi(t)} = \omega_0^2(t)\, e^{it}. \tag{2.9}$$

Moreover, we have

$$e^{i\varphi^{[2]}(t)} = e^{i\varphi(\varphi(t))} = \omega_0^2(\varphi(t))\, e^{i\varphi(t)} = \omega_1^2(t)\, \omega_0^2(t)\, e^{it} = (\omega_0(t)\, \omega_1(t))^2\, e^{it}$$

and

$$e^{i\varphi^{[m]}(t)} = (\omega_0(t)\, \omega_1(t) \dots \omega_{m-1}(t))^2\, e^{it}. \tag{2.10}$$

According to [2, Theorem 4.2], a convex annulus possesses Poncelet's porism property if and only if for some natural number $n \geq 3$ the function $\varphi^{[n]}(t) - t - 2\pi$ vanishes.

From this statement and the formula (2.10) we easily get the main theorem.

Theorem 2.1. *Let a circular annulus $\mathcal{A}$ possess Poncelet's porism property for a fixed natural number $n \geq 3$. Then the following identities hold*

$$(\omega_0\omega_1 \dots \omega_{n-1})^2 \equiv 1 \tag{2.11}$$

and

$$\operatorname{Im} \omega_0\omega_1 \dots \omega_{n-1} \equiv 0. \tag{2.12}$$

In the next sections we will show that Chapple's, Steiner's and other formulas (see [1], [4]) are consequences of the identity (2.12).

3. Bicentric triangle. If a triangle can be circuminscribed in a circular annulus $\mathcal{A}$, then the so-called Euler triangle formula holds

$$a^2 = \rho^2 - 2r\rho, \tag{3.1}$$

see [1], [4].

Let $\lambda_m = \lambda\left(\varphi^{[m]}(0)\right)$ for $m = 0, 1, 2$.

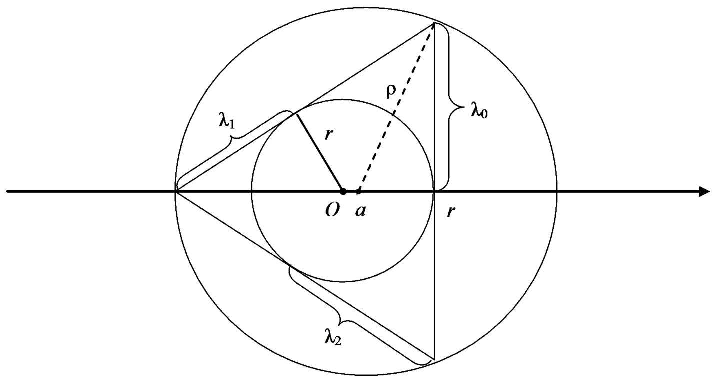

FIGURE 2.

Clearly, we see that

$$\lambda_0 = \lambda_2 = \sqrt{\rho^2 - (r-a)^2}\,, \qquad \lambda_1 = \sqrt{(\rho-a)^2 - r^2}. \tag{3.2}$$

From (2.12) we get

$$\begin{aligned} 0 &= \operatorname{Im} \omega_0(0)\,\omega_1(0)\,\omega_2(0) = \operatorname{Im} \omega_0^2(0)\,\omega_1(0) \\ &= \operatorname{Im}(r+\lambda_0 i)^2 (r+\lambda_1 i) = \left(r^2 - \lambda_0^2\right)\lambda_1 + 2r^2\lambda_0, \end{aligned}$$

i.e.,

$$\left(\lambda_0^2 - r^2\right)\lambda_1^2 = 4r^4\lambda_0^2.$$

Hence we have the following formulas, each of them being a consequence of the previous one.

$$\left[\rho^2 - (r-a)^2 - r^2\right]^2 \left[(\rho-a)^2 - r^2\right] = 4r^4\left[\rho^2 - (r-a)^2\right],$$

$$\begin{aligned} &\left[\rho^2 - (r-a)^2 - r^2\right]^2 (\rho-a)^2 \\ &\quad - \left[\left(\rho^2 - (r-a)^2\right)^2 - 2r^2\left(\rho^2 - (r-a)^2\right) + r^4\right] r^2 \\ &\quad = 4r^4\left[\rho^2 - (r-a)^2\right], \end{aligned}$$

$$\left[\rho^2 - (r-a)^2 - r^2\right]^2 (\rho - a)^2$$
$$= r^2 \left[\left(\rho^2 - (r-a)^2\right)^2 + 2r^2 \left(\rho^2 - (r-a)^2\right) + r^4\right],$$

$$\left[\rho^2 - (r-a)^2 - r^2\right]^2 (\rho - a)^2 = r^2 \left[\rho^2 - (r-a)^2 + r^2\right]^2,$$

$$\left[\rho^2 - (r-a)^2 - r^2\right] (\rho - a) = r \left[\rho^2 - (r-a)^2 + r^2\right],$$

$$\left(\rho^2 - (r-a)^2\right) (\rho - a) - r^2 (\rho - a) = r \left(\rho^2 - (r-a)^2\right) + r^3,$$

$$\left(\rho^2 - (r-a)^2\right) (\rho - a - r) = r^2 (\rho - a + r),$$

$$a^3 - (\rho + r) a^2 - \left(\rho^2 - 2r\rho\right) a + (\rho + r) \left(\rho^2 - 2r\rho\right) = 0,$$

$$(a - \rho - r) \left(a^2 - \rho^2 + 2r\rho\right) = 0$$

and the Euler triangle formula is proved.

4. Bicentric quadrileteral. If a 4-gon can be circuminscribed in a circular annulus $\mathcal{A}$, then the following formula holds

$$\left(\rho^2 - a^2\right)^2 = 2r^2 \left(\rho^2 + a^2\right), \tag{4.1}$$

see [1], [4].

Let $\lambda_m = \lambda\left(\varphi^{[m]}(0)\right)$ for $m = 0, 1, 2, 3$.

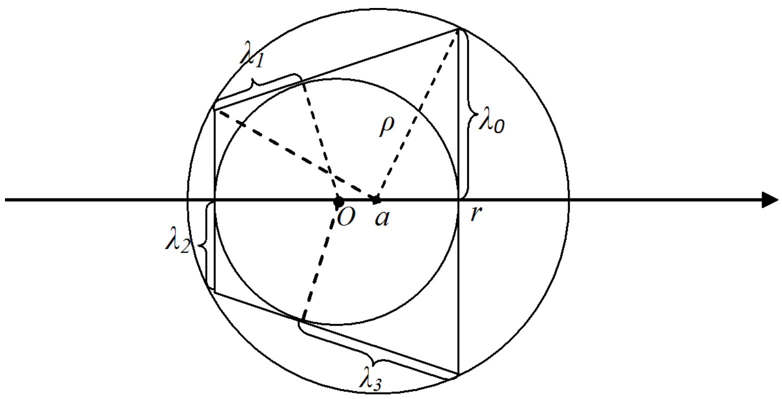

FIGURE 3.

It is easy to see that

$$\lambda_0 = \lambda_3 = \sqrt{\rho^2 - (r-a)^2}, \qquad \lambda_1 = \lambda_2 = \sqrt{\rho^2 - (r+a)^2}. \tag{4.2}$$

From (2.12) we have

$$\begin{aligned} 0 &= \operatorname{Im} \omega_0(0)\,\omega_1(0)\,\omega_2(0)\,\omega_3(0) = \operatorname{Im}(\omega_0(0)\,\omega_1(0))^2 \\ &= \operatorname{Im}((r+\lambda_0 i)(r+\lambda_1 i))^2 = \operatorname{Im}\left(r^2-\lambda_0\lambda_1 + r(\lambda_0+\lambda_1)\,i\right) \\ &= 2r(\lambda_0+\lambda_1)\left(r^2-\lambda_0\lambda_1\right) \end{aligned}$$

i.e.

$$r^2 = \lambda_0\lambda_1.$$

Hence we get

$$\begin{aligned} r^4 &= \left[\rho^2-(r-a)^2\right]\left[\rho^2-(r+a)^2\right], \\ r^4 &= \rho^4 - 2\left(r^2+a^2\right)\rho^2 + r^4 - 2r^2a^2 + a^4, \\ \rho^4 - 2a^2\rho^2 + a^4 &= 2r^2\rho^2 + 2r^2a^2. \end{aligned}$$

This is exactly the formula (4.1).

5. Bicentric pentagon. If a 5-gon can be circuminscribed in a circular annulus $\mathcal{A}$, then eight equivalent formulas are known (see [4, (43)–(50)]).

Let $\lambda_m = \lambda\left(\varphi^{[m]}(0)\right)$ for $m = 0, 1, 2, 3, 4$.

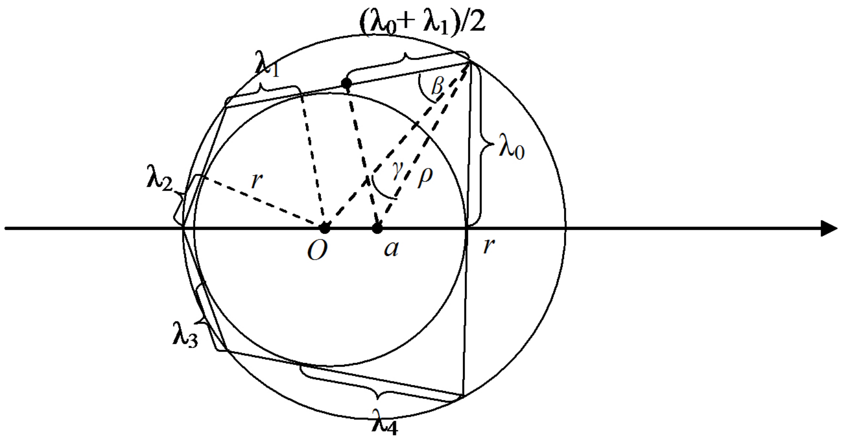

Figure 4.

It is easy to see that

$$(5.1) \quad \lambda_0 = \lambda_4 = \sqrt{\rho^2-(r-a)^2}, \quad \lambda_2 = \sqrt{(\rho-a)^2-r^2}, \quad \lambda_1 = \lambda_3.$$

We note that

$$a^2 = \lambda_0^2 + r^2 + \rho^2 - 2\rho\sqrt{\lambda_0^2+r^2}\cos\gamma,$$

$$\sin\beta = \frac{r}{\sqrt{\lambda_0^2+r^2}}, \qquad \cos\beta = \frac{\lambda_0}{\sqrt{\lambda_0^2+r^2}}, \qquad \frac{\lambda_0+\lambda_1}{2\rho} = \cos(\beta+\gamma).$$

Hence we obtain

$$\begin{aligned}\lambda_1 &= 2\rho\left[\cos\beta\cos\gamma - \sin\beta\sqrt{1-\cos^2\gamma}\right] - \lambda_0\\
&= 2\rho\left[\frac{\lambda_0}{\sqrt{\lambda_0^2+r^2}}\frac{\lambda_0^2+r^2+\rho^2-a^2}{2\rho\sqrt{\lambda_0^2+r^2}}\right.\\
&\qquad\left. - \frac{r}{\sqrt{\lambda_0^2+r^2}}\sqrt{1-\left(\frac{\lambda_0^2+r^2+\rho^2-a^2}{2\rho\sqrt{\lambda_0^2+r^2}}\right)^2}\right] - \lambda_0\\
&= \frac{1}{\lambda_0^2+r^2}\left[\lambda_0\left(\lambda_0^2+r^2+\rho^2-a^2\right)\right.\\
&\qquad\left. - r\sqrt{4\rho^2\left(\lambda_0^2+r^2\right)-\left(\lambda_0^2+r^2+\rho^2-a^2\right)^2}\right] - \lambda_0\\
&= \frac{1}{\lambda_0^2+r^2}\left[\lambda_0\left(\rho^2-a^2\right) - r\sqrt{4\rho^2\left(\lambda_0^2+r^2\right)-\left(\lambda_0^2+r^2+\rho^2-a^2\right)^2}\right]\\
&= \frac{1}{\lambda_0^2+r^2}\left[\lambda_0\left(\rho^2-a^2\right)-2ar\lambda_0\right],\end{aligned}$$

i.e.,

$$\lambda_1 = \frac{1}{\lambda_0^2+r^2}\lambda_0\left(\rho^2-a^2-2ar\right) = \lambda_0\frac{\rho^2-a^2-2ar}{\rho^2-a^2+2ar}. \tag{5.2}$$

From (2.12) we get

$$\begin{aligned}0 &= \operatorname{Im}\omega_0^2(0)\,\omega_1^2(0)\,\omega_2(0) = \operatorname{Im}\left((r+\lambda_0 i)(r+\lambda_1 i)\right)^2(r+\lambda_2 i)\\
&= \left[\left(r^2-\lambda_0\lambda_1\right)^2 - r^2\left(\lambda_0+\lambda_1\right)^2\right]\lambda_2 + 2r^2\left(r^2-\lambda_0\lambda_1\right)\left(\lambda_0+\lambda_1\right)\end{aligned}$$

or

$$\begin{aligned}0 &= \operatorname{Im}\omega_0^2(0)\,\omega_1^2(0)\,\omega_2(0) = \operatorname{Im}(r+\lambda_0 i)^2(r+\lambda_1 i)^2(r+\lambda_2 i)\\
&= \left[\left(r^2-\lambda_0^2\right)\left(r^2-\lambda_1^2\right)-4r^2\lambda_0\lambda_1\right]\lambda_2 + 2r^2\left[\left(r^2-\lambda_0^2\right)\lambda_1+\left(r^2-\lambda_1^2\right)\lambda_0\right].\end{aligned}$$

Now, it suffices to substitute λ_0, λ_2 and λ_1 given by (5.1) and (5.2), respectively. For $n \geq 6$ all calculations are similar.

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