doi: 10.2478/v10062-012-0002-4

ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. LXVI, NO. 1, 2012 SECTIO A 7–12

WALDEMAR CIEŚLAK and ELŻBIETA SZCZYGIELSKA

# Affine invariants of annuli

Abstract. A family of regular annuli is considered. Affine invariants of annuli are introduced.

**1. Introduction.** We denote by $\mathcal{C}$ a family of all plane, closed, strictly convex and regular curves (of the class $C^1$). It is well known [1], [4] that a curve $C \in \mathcal{C}$ can be parametrized by

$$z(t) = p(t)\, e^{it} + \dot{p}(t)\, ie^{it} \quad \text{for } t \in [0, 2\pi], \tag{1.1}$$

where $p$ is the support function of $C$ (the dot denotes the differentiation with respect to $t$). The tangent vector $\dot{z}(t)$ to $C$ at $z(t)$ is equal to

$$\dot{z}(t) = R(t)\, ie^{it}, \tag{1.2}$$

where the curvature radius $R$ of $C$ is given by the formula

$$R = p + \ddot{p} > 0. \tag{1.3}$$

We denote by $\Lambda$ a family of all $2\pi$-periodic, positive-valued functions $\lambda : \mathbf{R} \to \mathbf{R}$ of the class $C^1$.

In this paper we will consider a family $\mathcal{C}\Lambda$ of annuli. An annulus $CD$ is an element of $\mathcal{C}\Lambda$ if and only if

1° the inner curve $C$ belongs to $\mathcal{C}$,

2° the outer curve $D$ can be parametrized in the form

$$w(t) = z(t) + \lambda(t)\, ie^{it} \quad \text{for } t \in [0, 2\pi] \tag{1.4}$$

2000 *Mathematics Subject Classification.* 53A04.
*Key words and phrases.* Invariant, annulus.

with some function $\lambda \in \Lambda$.

We will use the differential equation

$$\lambda \dot{\eta} = R\eta - R \tag{1.5}$$

and its solution in the form

$$\eta(t,c) = 1 - c\exp\int_0^t \frac{R(m)}{\lambda(m)}dm \quad \text{for } t \in [0,2\pi], \tag{1.6}$$

where $c$ is an arbitrary constant.

**2. Invariants of annuli.** We note that

**Theorem 2.1.** *Let an annulus $CD$ belongs to $\mathcal{C}\Lambda$. The number $c_o(CD)$ given by the formula*

$$c_o(CD) = \exp\left(-\int_0^{2\pi} \frac{|\dot{z}(t)|}{\lambda(t)}dt\right) = \exp\left(-\int_0^{2\pi} \frac{R(t)}{\lambda(t)}dt\right) \tag{2.1}$$

*does not depend on parametrizations of $C$, $D$ and affine transformations.*

For the proof it suffices to note that $\dot{z}(t) = R(t)ie^{it}$ and $w(t) - z(t) = \lambda(t)ie^{it}$. It follows from (2.1) that

$$0 < c_o(CD) < 1. \tag{2.2}$$

Let $c_o = c_o(CD)$. If $c \in [0, c_o]$, then we have

$$0 < \eta(t,c) \le 1. \tag{2.3}$$

We consider a family of curves

$$\mathcal{V}(CD) = \{V(c) : 0 < c \le c_o\}, \tag{2.4}$$

where a curve $V(c)$ is given by the formula

$$v(t,c) = z(t) + \eta(t,c)\lambda(t)ie^{it} \quad \text{for } t \in [0,2\pi]. \tag{2.5}$$

Of course, curves of the family $\mathcal{V}(CD)$ are affine invariants. The inequality (2.3) implies that all curves of the family $\mathcal{V}(CD)$ are contained in the annulus $CD$ and $V(0) = D$. We have

$$v(0,c) - v(2\pi,c) = c\frac{1-c_o}{c_o}\lambda(0)i. \tag{2.6}$$

It follows from (2.6) and (2.2) that a curve $V(c)$ is not closed.

For a fixed curve $V(c)$ we have $v(0,c) = w(0) - c\lambda(0)i$ and $v(2\pi,c) = w(0) - \frac{c}{c_o}\lambda(0)i$. It is easy to see that the end point $v(2\pi,c)$ of $V(c)$ belongs to the segment joining points $w(0)$ and $v(0,c_o)$ if $c < c_o^2$. It means that if $c < c_o^2$, then the end point of $V(c)$ is the beginning point of another curve of the family $\mathcal{V}(CD)$.

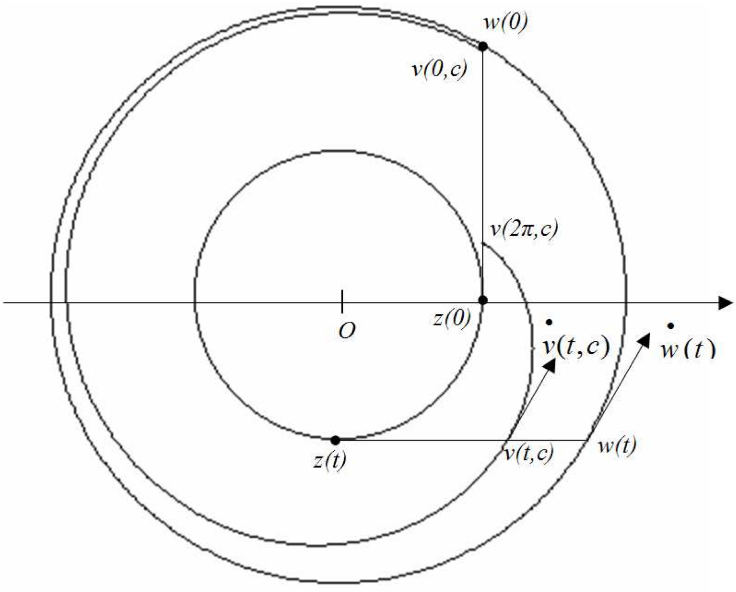


FIGURE 1

**Theorem 2.2.** *Let $CD \in \mathcal{C}\Lambda$ and $C$ be a curve of the class $C^2$. The following relations between tangent vectors and curvatures of $V(c)$ and $D$ hold*

$$\dot{v} = \eta \dot{w} \tag{2.7}$$

*and*

$$\eta k_{V(c)} = k_D. \tag{2.8}$$

**Proof.** Differentiating (2.5) and using the differential equation (1.5), we obtain

$$\dot{v} = \left(R + \dot{\eta}\lambda + \eta\dot{\lambda}\right) ie^{it} - \eta\lambda e^{it} = \eta\left(-\lambda e^{it} + \left(R + \dot{\lambda}\right) ie^{it}\right) = \eta\dot{w}.$$

Hence we obtain immediately (2.8). □

The following theorem explains a geometric meaning of the invariant $c_o$.

**Theorem 2.3.** *Let $CD \in \mathcal{C}\Lambda$. For an arbitrary curve $V(c) \in \mathcal{V}(CD)$ we have*

$$\left|\frac{v(2\pi, c) - v(0, c)}{v(0, c) - w(0)}\right| = \frac{1 - c_o}{c_o}, \tag{2.9}$$

*where $c_o = c_o(CD)$.*

**Proof.** We have

$$w(0) - v(0, c) = (1 - \eta(0, c))\, \lambda(0)\, i = c\lambda(0)\, i. \tag{2.10}$$

The formulas (2.6) and (2.10) imply (2.9). □

**Remark.** Theorem 2.3 is true if we take

$$\tilde{v}(t, c) = z(t) + \tilde{\eta}(t, c)\, \lambda(t)\, ie^{it} \quad \text{for } t \in [t_o, t_o + 2\pi],$$

where

$$\tilde{\eta}(t, c) = 1 - c \exp \int_{t_o}^{t} \frac{R(m)}{\lambda(m)} dm \quad \text{for } t \in [t_o, t_o + 2\pi].$$

**3. Estimation of $c_o$.** Let $C \in \mathcal{C}$. We fix $\lambda \in \Lambda$ and we denote by $C(\lambda)$ a curve given by the formula (1.4), i.e. $w(t) = z(t) + \lambda(t)\, ie^{it}$ for $t \in [0, 2\pi]$. Let

$$\lambda_m = \min_{[0,2\pi]} \lambda, \quad \lambda_M = \max_{[0,2\pi]} \lambda, \quad L(C) = \text{length } C. \tag{3.1}$$

The obvious inequality

$$\frac{L(C)}{\lambda_M} \leq \int_0^{2\pi} \frac{R(t)}{\lambda(t)} dt \leq \frac{L(C)}{\lambda_m}$$

implies the inequality for $c_o(CC(\lambda))$, namely

$$\exp\left(\frac{-L(C)}{\lambda_m}\right) \leq c_o(CC(\lambda)) \leq \exp\left(\frac{-L(C)}{\lambda_M}\right). \tag{3.2}$$

We note that

**Theorem 3.1.** *Let $A, B \in \mathcal{C}$ and $L(A) = L(B)$. If the function $\lambda \in \Lambda$ is constant, then*

$$c_o(AA(\lambda)) = c_o(BB(\lambda)). \tag{3.3}$$

**4. Special plane annuli.** Let $S_m$ denote the circle with the center at the origin and the radius $m$. We consider an annulus $S_r S_\rho$, where $\rho > r$. We have $\lambda(t) = \sqrt{\rho^2 - r^2}$, $R(t) = r$ and

$$c_o = c_o(S_r S_\rho) = \exp\left(\frac{-2\pi r}{\sqrt{\rho^2 - r^2}}\right). \tag{4.1}$$

Moreover, we have

$$\eta(t, c) = 1 - c \exp \frac{rt}{\sqrt{\rho^2 - r^2}} \tag{4.2}$$

and

$$v(t, c) = re^{it} + \left(1 - c \exp \frac{rt}{\sqrt{\rho^2 - r^2}}\right) \sqrt{\rho^2 - r^2} ie^{it} \tag{4.3}$$

for $t \in [0, 2\pi]$ and $c \in [0, c_o]$.

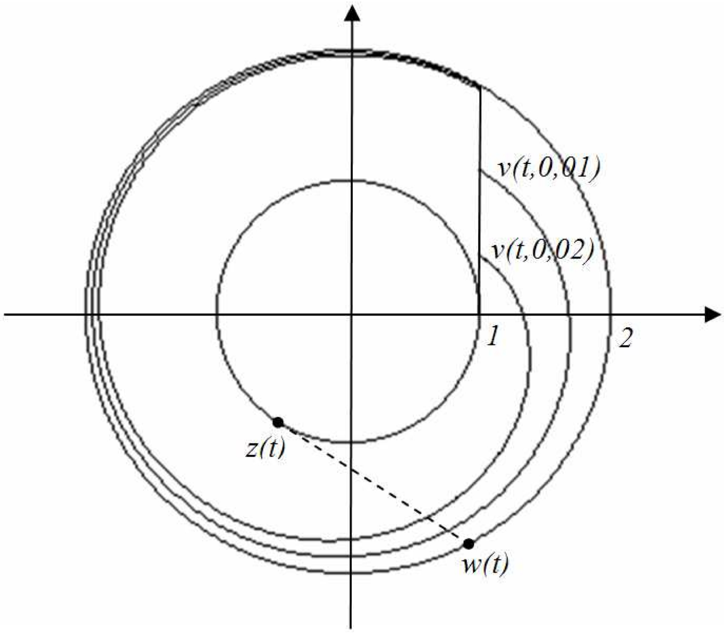


FIGURE 2

Two curves $v(t, c)$ given by (4.3) for $c = 0.01$ and $c = 0.02$ in a circular annulus formed by two concentric circles with $r = 1$ and $\rho = 2$ are presented in Figure 2.

**Theorem 4.1.** *Let $CD \in \mathcal{C}\Lambda$. We assume that $C$ is of the class $C^2$ and $D$ is a circle. The curvature $k_{V(c)}$ of a curve $V(c)$ is an increasing function.*

**Proof.** Let $t_2 > t_1$. The formulas (2.8) and (1.6) imply the inequality

$$\begin{aligned} k_{V(c)}(t_2) - k_{V(c)}(t_1) &= k_D \left( \frac{1}{\eta(t_2, c)} - \frac{1}{\eta(t_1, c)} \right) \\ &= \frac{k_D}{\eta(t_2, c)\, \eta(t_1, c)} c \left( \exp \int_0^{t_2} \frac{R(m)}{\lambda(m)} dm - \exp \int_0^{t_1} \frac{R(m)}{\lambda(m)} dm \right) > 0, \end{aligned}$$

where $c \in (0, c_0)$. Thus the curvature $k_{V(c)}$ is an increasing function. □

Let $C_\alpha$ be an $\alpha$-isoptic of $C \in \mathcal{C}$. We recall that an $\alpha$-isoptic $C_\alpha$ of $C$ consists of those points in the plane from which the curve is seen under the fixed angle $\pi - \alpha$, see [2], [3]. $C_\alpha$ has the form

(4.4) $z_\alpha(t) = z(t) + \lambda(t, \alpha)\, i e^{it} = z(t, \alpha) + \mu(t, \alpha)\, i e^{i(t+\alpha)} \quad \text{for } t \in [0, 2\pi],$

where

$$\lambda(t,\alpha) = \frac{1}{\sin\alpha}\left[p(t+\alpha) - p(t)\cos\alpha - \dot{p}(t)\sin\alpha\right] \tag{4.5}$$

and

$$\mu(t,\alpha) = \frac{1}{\sin\alpha}\left[p(t+\alpha)\cos\alpha - \dot{p}(t+\alpha)\sin\alpha - p(t)\right] < 0. \tag{4.6}$$

Moreover, we have

$$\frac{\partial\lambda}{\partial\alpha} = \frac{-\mu}{\sin\alpha} > 0, \tag{4.7}$$

see [3].

We consider a family of all annuli $CC_\alpha$ and the function

$$c_o(\alpha) = c_o(CC_\alpha) = \exp\left(-\int_0^{2\pi} \frac{R(t)}{\lambda(t,\alpha)}dt\right) \quad \text{for } \alpha \in (0,\pi). \tag{4.8}$$

With respect to (4.8) we have

$$\frac{d}{d\alpha}\int_0^{2\pi} \frac{R(t)}{\lambda(t,\alpha)}dt = \int_0^{2\pi} \frac{R(t)\,\mu(t,\alpha)}{\lambda^3(t,\alpha)}dt < 0.$$

Hence and from the definition of $c_o(\alpha)$ it follows immediately that the mapping $\alpha \to c_o(\alpha)$ is strictly increasing.

## References


[1] Bonnesen, T., Fenchel, W., *Theorie der konvexen Körper*, Chelsea Publishing Co., New York, 1948.

[2] Cieślak, W., Miernowski, A. and Mozgawa, W., *Isoptics of a closed strictly convex curve*, Global differential geometry and global analysis (Berlin, 1990), Lecture Notes in Math., **1481**, Springer, Berlin, 1991, 28–35.

[3] Cieślak, W., Miernowski, A. and Mozgawa, W., *Isoptics of a closed strictly convex curve. II*, Rend. Sem. Mat. Univ. Padova, **96** (1996), 37–49.

[4] Santalo, L., *Integral geometry and geometric probability*, Encyclopedia of Mathematics and its Applications, Vol. 1. Addison-Wesley Publishing Co., Reading, Mass.–London–Amsterdam, 1976.



Waldemar Cieślak
Politechnika Lubelska
Zakład Matematyki
ul. Nadbystrzycka 40
20-618 Lublin
Poland
e-mail: `izacieslak@wp.pl`

Elżbieta Szczygielska
Państwowa Wyższa Szkoła Zawodowa
w Białej Podlaskiej
ul. Sidorska 95/97
21-500 Biała Podlaska
Poland
e-mail: `eszczygielska@o2.pl`


Received January 8, 2011