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# On strong proximinality in normed linear spaces

Abstract. The paper deals with strong proximinality in normed linear spaces. It is proved that in a compactly locally uniformly rotund Banach space, proximinality, strong proximinality, weak approximative compactness and approximative compactness are all equivalent for closed convex sets. How strong proximinality can be transmitted to and from quotient spaces has also been discussed.

**1. Introduction.** Let $W$ be a closed subset of a normed linear space $(X, \|.\|)$. The *metric projection* of $X$ onto $W$ is the set-valued map $P_W$ defined by $P_W(x) = \{y \in W : \|x - y\| \le \|x - w\| \ for \ all \ w \in W\}$. The set $W$ is said to be *proximinal* (*Chebyshev*) if for every $x \in X$, $P_W(x)$ is non-empty (a singleton).

A stronger form of proximinality, called strong proximinality by Godefroy and Indumathi [6] has been discussed by several researchers (see e.g. [1], [3], [5]–[8] and references cited therein). Vlasov [11] has also studied this concept under the name H-sets.

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A proximinal subset $W$ of a normed linear space $(X, \|.\|)$ is said to be *strongly proximinal* at $x \in X$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for every $y \in P_W(x, \delta)$ there is some $y' \in P_W(x)$ satisfying $\|y - y'\| < \varepsilon$ or equivalently, $P_W(x, \delta) \subseteq P_W(x) + \varepsilon B_X$, where $P_W(x, \delta) = \{y \in W : \|x - y\| < d(x, W) + \delta\}$ and $B_X$ is the unit ball in $X$. The set $W$ is said to be strongly proximinal in $X$ if it is strongly proximianl at all points of $X$.

A proximinal set need not be strongly proximinal (see [3]), even a Chebyshev set need not be strongly proximinal (see [8]).

A subset $W$ of a normed linear space $X$ is said to be *approximatively compact* [4] (*weakly approximatively compact*) for $x \in X$, if every minimizing sequence $\{y_n\} \subseteq W$ for $x$, i.e. $\|x - y_n\| \to d(x, W)$, has a convergent subsequence in $W$ (a weakly convergent subsequence in $W$).

A subset $W$ of a normed linear space $X$ is said to be *strongly Chebyshev* [1] for $x \in X$, if every minimizing sequence $\{y_n\} \subseteq W$ for $x$ is convergent in $W$.

The set $W$ is said to be approximatively compact or weakly approximatively compact or strongly Chebyshev in $X$ if it is so at every point $x \in X$.

It is known (see [1]) that approximatively compact sets are strongly proximinal and strongly Chebyshev sets are precisely the sets which are strongly proximinal and Chebyshev.

A normed linear space $X$ is said to be *locally uniformly rotund* (*LUR*) if for every $x \in S_X = \{x \in X : \|x\| = 1\}$ and every sequence $\{x_n\}$ in $S_X$ satisfying $\|x + x_n\| \to 2$, we have $x_n \to x$.

A normed linear space $X$ is said to be *compactly locally uniformly rotund* (*CLUR*) if for every $x \in S_X$ and every sequence $\{x_n\}$ in $S_X$ satisfying $\|x + x_n\| \to 2$, the sequence $\{x_n\}$ has a convergent subsequence.

A normed linear space $X$ is said to be *compactly weakly locally uniformly rotund* (*CWLUR*) if for every $x \in S_X$ and every sequence $\{x_n\}$ in $S_X$ satisfying $\|x+x_n\| \to 2$, the sequence $\{x_n\}$ has a weakly convergent subsequence.

A normed linear space $X$ is said to have *property (H)* if for any $\{x_n\} \subseteq S_X$ and $x \in S_X$ satisfying $x_n \to x$ weakly, we have $x_n \to x$.

Clearly, every finite-dimensional normed linear space is CLUR, and LUR normed linear spaces are CLUR. It is easy to prove (see [9]) that a normed linear space is LUR if and only if it is rotund and CLUR. Moreover, CLUR spaces have property (H).

In this paper, we prove some results concerning strong proximinality in normed linear spaces. We see how strong proximinality can be transmitted to and from quotient spaces, and prove that for a proximinal linear subspace $M$ of a normed linear space $X$, if $W \supseteq M$ is strongly Chebyshev in $X$ then $W/M$ is also strongly Chebyshev in $X/M$. We also prove that in a CLUR Banach space, proximinality, strong proximinality, weak approximative compactness and approximative compactness are equivalent for closed convex sets.

**2. Main results.** We start with proving some basic results concerning strong proximinality.

**Proposition 2.1.** *Let $W$ be a linear subspace of a normed linear space $(X, \|\cdot\|)$. If $W$ is strongly proximinal at $x \in X$, then $W + y$ is strongly proximinal at $x + y$ for all $y \in X$.*

**Proof.** Let $\varepsilon > 0$ be given. Since $W$ is strongly proximinal at $x$, there exists a $\delta > 0$ such that $P_W(x, \delta) \subseteq P_W(x) + \varepsilon B_X$. This implies that $P_W(x, \delta) + y \subseteq [P_W(x) + \varepsilon B_X] + y$ *for all* $y \in X$, i.e., $P_{W+y}(x + y, \delta) \subseteq P_{W+y}(x + y) + \varepsilon B_X$ for all $y \in X$. Hence $W + y$ is strongly proximinal at $x + y$. $\square$

**Proposition 2.2.** *Let $W$ be a linear subspace of a normed linear space $(X, \|\cdot\|)$. If $W$ is strongly proximinal at $x$, then $W$ is strongly proximinal at $\alpha x$ for every scalar $\alpha$.*

**Proof.** Suppose $\alpha = 0$, then $\alpha x = 0$. As $0 \in W$, $P_W(0) = \{0\}$. For any $\varepsilon > 0$, take $\delta = \varepsilon$. Then the inclusion $P_W(0, \varepsilon) \subseteq P_W(0) + \varepsilon B_X$ implies that $W$ is strongly proximinal at 0. Now, suppose $\alpha \neq 0$. Let $\varepsilon > 0$ be arbitrary and $x \in X$. Since $W$ is strongly proximinal at $x$, for $\frac{\varepsilon}{|\alpha|} > 0$, there exists some $\delta_1 > 0$ such that for every $y \in P_W(x, \delta_1)$ there is some $y' \in P_W(x)$ satisfying $\|y - y'\| < \frac{\varepsilon}{|\alpha|}$.

Let $\delta = |\alpha|\delta_1$ and $z \in P_W(\alpha x, \delta)$, then $\|\alpha x - z\| < \|\alpha x - w\| + \delta$ for all $w \in W$. This implies that $\|x - \frac{z}{|\alpha|}\| < \|x - w'\| + \frac{\delta}{|\alpha|}$ for all $w' \in W$, i.e., $\frac{z}{|\alpha|} \in P_W(x, \delta_1)$. Since $W$ is strongly proximinal at $x$, there exists $z' \in P_W(x)$ satisfying $\|\frac{z}{|\alpha|} - z'\| < \frac{\varepsilon}{|\alpha|}$. Then for any $z \in P_W(\alpha x, \delta)$ there exists $z' \in P_W(x)$, i.e., $\alpha z' \in P_W(\alpha x)$ satisfying $\|z - \alpha z'\| = |\alpha| \|\frac{z}{|\alpha|} - z'\| < \varepsilon$. Therefore, $W$ is strongly proximinal at $\alpha x$ for $\alpha \neq 0$ and hence for every scalar $\alpha$. $\square$

It is known (see [10]) that if $W$ is a Chebyshev subset of a normed linear space $(X, \|\cdot\|)$, then $P_W(x) = P_W(\alpha x + (1 - \alpha)P_W(x))$ for every scalar $\alpha \in [0, 1]$. Using this property, we show that a similar result is true for strong proximinality.

**Theorem 2.3.** *Let $W$ be a Chebyshev subset of a normed linear space $(X, \|\cdot\|)$. If $W$ is strongly proximinal at $x$, then $W$ is strongly proximinal at $\alpha x + (1 - \alpha)P_W(x)$ for every scalar $\alpha \in [0, 1]$.*

**Proof.** Let $\varepsilon > 0$ be arbitrary and $x \in X$. Since $W$ is strongly proximinal at $x$, there exists a $\delta_1 > 0$ such that for every $y \in P_W(x, \delta_1)$ there is some $y' \in P_W(x)$ satisfying $\|y - y'\| < \varepsilon$.

Let $z = \alpha x + (1 - \alpha)P_W(x)$, $0 \leq \alpha \leq 1$ and $P_W(x) = \{y'\}$. Then $P_W(z) = \{y'\}$ and

$$\|x - z\| + \|z - y'\| = \|x - y'\|. \tag{2.1}$$

Suppose $z' \in P_W(z, \delta)$, $\delta = \delta_1$. Then

$$\|z - z'\| < \|z - w\| + \delta \tag{2.2}$$

for all $w \in W$.

We claim that $z' \in P_W(x, \delta)$. Using (2.2), we obtain

$$\|x - z'\| \leq \|x - z\| + \|z - z'\| < \|x - z\| + \|z - w\| + \delta$$

for all $w \in W$, i.e. $\|x - z'\| < \|x - z\| + \|z - y'\| + \delta$, as $y' \in W$. By (2.1), this gives

$$\|x - z'\| < \|x - y'\| + \delta,$$

i.e., $\|x - z'\| < \|x - w\| + \delta$ for all $w \in W$, as $y' \in P_W(x)$. Therefore, $z' \in P_W(x, \delta)$. Since $W$ is strongly proximinal at $x$, for $\{y'\} = P_W(x)$, we have $\|z' - y'\| < \varepsilon$. Thus for any $z' \in P_W(z, \delta)$ there exists $\{y'\} = P_W(z)$ satisfying $\|z' - y'\| < \varepsilon$. Hence $W$ is strongly proximinal at $z = \alpha x + (1 - \alpha)P_W(x)$ for every scalar $\alpha$, $0 \leq \alpha \leq 1$. □

Concerning the strong proximinality in quotient spaces, we have the following result.

**Theorem 2.4.** *Let $M$ be a closed linear subspace of a normed linear space $(X, \|\cdot\|)$ and $W$ a linear subspace of $X$ such that $W \supseteq M$. If $W$ is strongly proximinal at $x$, then $W/M$ is strongly proximinal at $x + M$.*

**Proof.** Let $\varepsilon > 0$ be arbitrary and $x \in X$. Since $W$ is strongly proximinal at $x$, there exists a $\delta_1 > 0$ such that for every $y \in P_W(x, \delta_1)$ there is some $y' \in P_W(x)$ satisfying $\|y - y'\| < \varepsilon$.

Let $z + M \in P_{W/M}(x + M, \delta)$, $\delta = \delta_1$. Then

$$\|(x + M) - (z + M)\| < \|(x + M) - (w + M)\| + \delta$$

for all $w + M \in W/M$. This implies

$$\inf_{m \in M} \|(x - z) - m\| < \|x - w\| + \delta$$

for all $w \in W$. Then there exists $m' \in M$ such that

$$\|(x - z) - m'\| < \|x - w\| + \delta$$

for all $w \in W$.

This gives $z + m' \in P_W(x, \delta)$. Since $W$ is strongly proximinal at $x$, there exists $z' \in P_W(x)$ satisfying $\|(z + m') - z'\| < \varepsilon$. Also $z' \in P_W(x)$ gives $z' + M \in P_{W/M}(x + M)$ (see [2]). Therefore,

$$\|(z + M) - (z' + M)\| = \inf_{m \in M} \|(z - z') - m\| \leq \|(z - z') + m'\| < \varepsilon.$$

Hence $W/M$ is strongly proximinal at $x + M$. □

**Remarks.** (i) If $M$ is a closed linear subspace of a normed linear space $X$ and $W \supseteq M$ is a strongly proximinal subspace in $X$, then $W/M$ is strongly proximinal in $X/M$.
(ii) The authors do not know whether the converse of Theorem 2.4 hold? However, it was proved in [8] that if $M$ is an infinite dimensional proximinal Banach space, then $M$ can be embedded isometrically as a nonstrongly proximinal hyperplane in another Banach space $W$. Thus, $\dim W/M = 1$ and so it is strongly proximinal in all its super spaces (see [8]). Then $W/M$ is proximinal in all its super spaces and so $W$ is proximinal in all its super spaces (see [2]). Using the same technique, $W$ can be embedded as a nonstrongly proximinal hyperplane in another Banach space.

We require the following lemma given in [2] for our next result.

**Lemma 2.5.** *Let $M$ be a proximinal subspace of a normed linear space $(X, \|.\|)$ and $W$ a linear subspace of $X$ such that $W \supseteq M$. If $W$ is Chebyshev in $X$, then $W/M$ is Chebyshev in $X/M$.*

Using the above lemma and Theorem 2.4, we obtain the following theorem.

**Theorem 2.6.** *Let $M$ be a proximinal linear subspace of a normed linear space $(X, \|\cdot\|)$ and $W$ a linear subspace of $X$ such that $W \supseteq M$. If $W$ is strongly Chebyshev in $X$, then $W/M$ is strongly Chebyshev in $X/M$.*

It is well known (see [1]) that a Banach $X$ is reflexive if and only if every closed convex subset of $X$ is proximinal or if and only if every closed convex subset of $X$ is weakly approximatively compact. Analogously, the following result shows that in a CLUR Banach space $X$, a closed convex set is proximinal if and only if it is weakly approximatively compact or if and only if it is strongly proximinal.

**Theorem 2.7.** *Let $W$ be a closed convex subset of a CLUR Banach space $(X, \|\cdot\|)$ then the following are equivalent:*
(i) *$W$ is proximinal.*
(ii) *$W$ is weakly approximatively compact.*
(iii) *$W$ is approximatively compact.*
(iv) *$W$ is strongly proximinal.*

**Proof.** (i) $\Rightarrow$ (ii) Let $x \in X$ be arbitrary. If $x \in W$, then the result is obvious, so suppose $x \in X \backslash W$. Without loss of generality, we may assume that $x = 0$. Let $y \in P_W(0)$ and $\delta = d(0, W)$. Suppose that $\{x_n\}$ is a minimizing sequence in $W$ for 0, i.e.,

$$\lim_{n\to\infty} \|x_n\| = d(0, W) = \delta.$$

Notice that

$$\delta \leq \left\| \frac{x_n + y}{2} \right\| \leq \frac{\|x_n\| + \|y\|}{2} \to \delta. \tag{2.3}$$

For every $n \in \mathbb{N}$, put $p_n = \delta \frac{x_n}{\|x_n\|}$. Then

$$\left\|\frac{p_n + y}{2}\right\| = \left\|\frac{\delta x_n + y\|x_n\|}{2\|x_n\|}\right\|.$$

Using (2.3), we have

$$\lim_{n\to\infty}\left\|\frac{p_n + y}{2}\right\| = \lim_{n\to\infty}\left\|\frac{\delta x_n + y\|x_n\|}{2\|x_n\|}\right\| = \delta.$$

Since $X$ being CLUR is CWLUR, $\{p_n\}$ has a weakly convergent subsequence $p_{n_i} \to p$ weakly. This gives $x_{n_i} \to p$ weakly and hence $W$ is weakly approximatively compact.

(ii) $\Rightarrow$ (iii) Let $\{y_n\} \subseteq W$ be any minimizing sequence for $x \in X\backslash W$, i.e.,

$$\lim_{n\to\infty} \|x - y_n\| = d(x, W).$$

Since $W$ is weakly approximatively compact, $\{y_n\}$ has a subsequence $y_{n_i} \to y$ weakly. Since $W$ is closed and convex $y \in W$. Based on the weak lower semi-continuity of the norm, we get

$$\|x - y\| \le \liminf_{i\to\infty} \|x - y_{n_i}\| = d(x, W),$$

i.e., $y \in P_W(x)$). Therefore,

$$\|x - y\| = d(x, W) = \lim_{i\to\infty} \|x - y_{n_i}\|. \tag{2.4}$$

Also as $y_{n_i} \to y$ weakly, we have $x - y_{n_i} \to (x - y)$ weakly. Since $X$ is CLUR, it has property (H). Therefore using (2.4), we obtain

$$\|y - y_{n_i}\| = \|(x - y_{n_i}) - (x - y)\| \to 0.$$

Hence $W$ is approximatively compact.

(iii) $\Rightarrow$ (iv) is proved in [1].

(iv) $\Rightarrow$ (i) is obvious. $\square$

**Remark.** If $W$ is a closed convex subset of a LUR Banach space $X$, then the proximinality of $W$ implies that every minimizing sequence in $W$ is convergent.

Since for a closed convex subset of a LUR Banach space, best approximation if it exist, is always unique, we obtain

**Corollary 2.8.** *Let $W$ be a closed convex subset of a LUR Banach space $X$ then the following statements are equivalent:*

(i) *$W$ is weakly approximatively compact.*

(ii) *$W$ is approximatively compact.*

(iii) *$W$ is strongly proximinal.*

(iv) *$W$ is strongly Chebyshev.*

(v) *$W$ is Chebyshev.*

In general, strong proximinality need not imply approximative compactness.

**Example 2.9.** Let $X = l_\infty$, $W = c_0$. Then $W$ being an M-ideal is strongly proximinal (see [5]) in $X$. But, for $x = (1, 1, 1, \dots) \in l_\infty$, the sequence $y_n = (1, 1, \dots, 1, 0, 0, \dots) \in W$ is minimizing sequence for $x$ but $\{y_n\}$ has no convergent subsequence.

Analogous to Theorem 2.7, we have the following result.

**Theorem 2.10.** *Let $W$ be a closed convex subset of a CWLUR Banach space $(X, \|\cdot\|)$ then the following are equivalent:*
(i) *$W$ is proximinal.*
(ii) *$W$ is weakly approximatively compact.*

**Proof.** (i) $\Rightarrow$ (ii) The proof runs on similar lines as that of Theorem 2.7. (ii) $\Rightarrow$ (i) is proved in [12]. □

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