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An integral operator on the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{CVH}(\beta)$

Abstract. The purpose of this paper is to study some properties related to convexity order and coefficients estimation for a general integral operator. We find the convexity order for this operator, using the analytic functions from the class of starlike functions of order α and from the class $\mathcal{CVH}(\beta)$ and also we estimate the first two coefficients for functions obtained by this operator applied on the class $\mathcal{CVH}(\beta)$.

1. Preliminary and definitions. We consider the class of analytic functions $f(z)$, in the open unit disk, $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$, having the form:

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \quad z \in \mathcal{U}. \tag{1.1}$$

This class is denoted by $\mathcal{A}$. By $\mathcal{S}$ we denote the class of all functions from $\mathcal{A}$ which are univalent in $\mathcal{U}$.

We denote by $\mathcal{K}(\alpha)$ the class of all convex functions of order α $(0 \le \alpha < 1)$ that satisfy the inequality:

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > \alpha, \quad z \in \mathcal{U}.$$

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A function $f \in \mathcal{A}$ is in the class $\mathcal{S}^*(\alpha)$, of starlike functions of order α if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in \mathcal{U}.$$

These classes were introduced by Robertson in [4] and studied by many other authors.

We also consider the class $\mathcal{CVH}(\beta)$ which was introduced by Acu and Owa in [1]. An analytic function f is in the class $\mathcal{CVH}(\beta)$ with $\beta > 0$ if we have the following inequality:

$$(1.2) \quad \left|\frac{zf''(z)}{f'(z)} - 2\beta(\sqrt{2}-1) + 1\right| < \operatorname{Re}\left(\sqrt{2}\frac{zf''(z)}{f'(z)}\right) + 2\beta(\sqrt{2}-1) + \sqrt{2},$$

where $z \in \mathcal{U}$.

Remark 1. This class is well defined for $\operatorname{Re}\left(\sqrt{2}\frac{zf''(z)}{f'(z)}\right) > 2\beta(1-\sqrt{2})-\sqrt{2}$.

For this class the following result was proved by Acu and Owa in [1].

Theorem 1.1. *If $f(z) = z+\sum_{j=2}^{\infty} a_j z^j$ belongs to the class $\mathcal{CVH}(\beta)$, $\beta > 0$, then*

$$|a_2| \le \frac{1+4\beta}{2(1+2\beta)}, \quad |a_3| \le \frac{(1+4\beta)(3+16\beta+24\beta^2)}{12(1+2\beta)^3}.$$

For the analytic functions f_i and g_i we consider the operator

$$(1.3) \qquad K(z) = \int_0^z \prod_{i=1}^{n} (g_i'(t))^{\eta_i} \cdot \left(\frac{f_i(t)}{t}\right)^{\gamma_i} dt,$$

for $\gamma_i, \eta_i > 0$ with $i = \overline{1,n}$. This operator was studied by Pescar in [3] and Ularu in [5].

We study the properties of this operator on the classes $\mathcal{CVH}(\beta)$ and $\mathcal{S}^*(\alpha)$. The idea of this paper was given by an open problem considered by N. Breaz, D. Breaz and Acu in [2].

2. Main results. Let

$$\phi = 1 - \sum_{i=1}^{n} \eta_i - (2-\sqrt{2})\sum_{i=1}^{n} \eta_i\beta_i + \sum_{i=1}^{n} \gamma_i(\alpha_i - 1),$$

where $\beta_i > 0$, $\alpha_i \in [0,1)$ and $\eta_i, \gamma_i > 0$ for all $i = \overline{1,n}$. For

$$(2.1) \qquad \sum_{i=1}^{n} \eta_i + (2-\sqrt{2})\sum_{i=1}^{n} \eta_i\beta_i + \sum_{i=1}^{n} \gamma_i(\alpha_i - 1) \le 1$$

we have that $0 \le \phi < 1$.

Theorem 2.1. *If $f_i \in \mathcal{S}^*(\alpha_i)$ and $g_i \in \mathcal{CVH}(\beta_i)$, with $\beta_i > 0$, $0 \leq \alpha_i < 1$ and $\eta_i, \gamma_i > 0$ for all $i = \overline{1,n}$ satisfying the condition (2.1), then the integral operator $K(z)$ defined by (1.3) is in the class $\mathcal{K}(\phi)$, $0 \leq \phi < 1$ where*

$$\phi = 1 - \sum_{i=1}^{n} \eta_i - (2-\sqrt{2}) \sum_{i=1}^{n} \eta_i \beta_i + \sum_{i=1}^{n} \gamma_i(\alpha_i - 1).$$

Proof. From the definition of $K(z)$ we obtain:

$$\frac{zK''(z)}{K'(z)} = \sum_{i=1}^{n} \left(\eta_i \frac{zg_i''(z)}{g_i'(z)} \right) + \sum_{i=1}^{n} \left[\gamma_i \left(\frac{zf_i'(z)}{f_i(z)} - 1 \right) \right].$$

Further we have:

$$\begin{aligned} \sqrt{2}\mathrm{Re}\left(\frac{zK''(z)}{K'(z)} + 1 \right) &= \mathrm{Re} \sum_{i=1}^{n} \sqrt{2}\eta_i \frac{zg_i''(z)}{g_i'(z)} \\ &\quad + \sqrt{2} + \sqrt{2}\mathrm{Re} \sum_{i=1}^{n} \gamma_i \frac{zf_i'(z)}{f_i(z)} - \sqrt{2}\mathrm{Re} \sum_{i=1}^{n} \gamma_i. \end{aligned}$$

We use the fact that f_i are starlike functions of order α_i and $g_i \in \mathcal{CVH}(\beta_i)$ for $i = \overline{1,n}$:

$$\begin{aligned} \sqrt{2}\mathrm{Re}\left(\frac{zK''(z)}{K'(z)} + 1 \right) &> \sum_{i=1}^{n} \eta_i \left| \frac{zg_i''(z)}{g_i'(z)} - 2\beta_i(\sqrt{2}-1) + 1 \right| \\ &\quad - \sum_{i=1}^{n} \left(2\eta_i\beta_i(\sqrt{2}-1) + \eta_i\sqrt{2} \right) + \sqrt{2} \\ &\quad + \sqrt{2} \sum_{i=1}^{n} \gamma_i \alpha_i - \sqrt{2} \sum_{i=1}^{n} \gamma_i \\ &> -\sqrt{2} \sum_{i=1}^{n} \eta_i - 2(\sqrt{2}-1) \sum_{i=1}^{n} \eta_i\beta_i + \sqrt{2} \\ &\quad + \sqrt{2} \sum_{i=1}^{n} \gamma_i \alpha_i - \sqrt{2} \sum_{i=1}^{n} \gamma_i. \end{aligned}$$

From these inequalities we obtain that:

$$\mathrm{Re}\left(\frac{zK''(z)}{K'(z)} + 1 \right) > 1 - \sum_{i=1}^{n} \eta_i - (2-\sqrt{2}) \sum_{i=1}^{n} \eta_i\beta_i + \sum_{i=1}^{n} \gamma_i(\alpha_i - 1).$$

So we obtain the convexity order for the operator $K(z)$ for functions in the classes $\mathcal{S}^*(\alpha_i)$ and $\mathcal{CVH}(\beta_i)$ for $i = \overline{1,n}$. $\square$

For $\eta_1 = \eta_2 = \cdots = \eta_n = 1$ and $\gamma_1 = \gamma_2 = \cdots = \gamma_n = 1$ in the definition of $K(z)$ given by (1.3) we obtain:

$$K_1(z) = \int_0^z \prod_{i=1}^n g_i'(t) \cdot \frac{f_i(t)}{t} dt$$

for $i = \overline{1,n}$.

Corollary 2.2. *If $f_i \in \mathcal{S}^*(\alpha_i)$ and $g_i \in \mathcal{CVH}(\beta_i)$, for $\beta_i > 0$, $0 \le \alpha_i < 1$ for all $i = \overline{1,n}$, then the integral operator*

$$K_1(z) = \int_0^z \prod_{i=1}^n g_i'(t) \cdot \frac{f_i(t)}{t} dt$$

is convex of order ϕ, where

$$\phi = 1 - n - (2 - \sqrt{2}) \sum_{i=1}^n \beta_i + \sum_{i=1}^n (\alpha_i - 1),$$

for $0 \le \phi < 1$.

Next we will obtain the estimation for the coefficients of the operator $K_1(z)$ defined above.

Theorem 2.3. *Let $f_i \in \mathcal{CVH}(\gamma_i)$, $g_i \in \mathcal{CVH}(\beta_i)$, with $\beta_i, \gamma_i > 0$ and $g_i(z) = z + \sum_{j=2}^\infty a_{i,j} z^j$, $f_i(z) = z + \sum_{j=2}^\infty b_{i,j} z^j$ for all $i = \overline{1,n}$. If $K_1(z) = z + \sum_{j=2}^\infty c_j z^j$, then we obtain:*

$$|c_2| \le \frac{1}{2} \left(\sum_{i=1}^n \frac{1 + 4\gamma_i}{2(1 + 2\gamma_i)} + \sum_{i=1}^n \frac{1 + 4\beta_i}{1 + 2\beta_i} \right)$$

and

$$\begin{aligned}
|c_3| \le \frac{1}{3} & \left[\sum_{i=1}^n \frac{(1 + 4\gamma_i)(3 + 16\gamma_i + 24\gamma_i^2)}{12(1 + 2\gamma_i)^3} \right. \\
& \left. + \sum_{k=1}^{n-1} \left(\frac{1 + 4\gamma_k}{2(1 + 2\gamma_k)} \sum_{i=k+1}^n \frac{1 + 4\gamma_i}{2(1 + 2\gamma_i)} \right) \right] \\
& + \sum_{i=1}^n \frac{(1 + 4\beta_i)(3 + 16\beta_i + 24\beta_i^2)}{12(1 + 2\beta_i)^3} \\
& + \frac{2}{3} \left[2 \sum_{k=1}^{n-1} \left(\frac{1 + 4\beta_k}{2(1 + 2\beta_k)} \sum_{i=k+1}^n \frac{1 + 4\beta_i}{2(1 + 4\beta_i)} \right) \right. \\
& \left. + \left(\sum_{i=1}^n \frac{1 + 4\beta_i}{2(1 + 2\beta_i)} \right) \left(\sum_{i=1}^n \frac{1 + 4\gamma_i}{2(1 + 2\gamma_i)} \right) \right].
\end{aligned}$$

Proof. From the definition of $K_1(z)$ we obtain:

$$K_1'(z) = \prod_{i=1}^{n} g_i'(z) \cdot \frac{f_i(z)}{z}$$

and further we get that:

$$\begin{aligned} 1 + \sum_{j=2}^{\infty} j c_j z^{j-1} &= \left(1 + \sum_{j=2}^{\infty} j a_{1,j} z^{j-1}\right) \ldots \left(1 + \sum_{j=2}^{\infty} j a_{n,j} z^{j-1}\right) \\ &\quad \times \left(1 + \sum_{j=2}^{\infty} b_{1,j} z^{j-1}\right) \ldots \left(1 + \sum_{j=2}^{\infty} b_{n,j} z^{j-1}\right). \end{aligned}$$

After some computation from the above relation we obtain:

$$c_2 = \frac{1}{2} \sum_{i=1}^{n} b_{i,2} + \sum_{i=1}^{n} a_{i,2} \tag{2.2}$$

and

$$\begin{aligned} c_3 &= \frac{1}{3} \sum_{i=1}^{n} b_{i,3} + \sum_{i=1}^{n} a_{i,3} + \frac{1}{3} \sum_{k=1}^{n-1} \left(b_{k,2} \sum_{i=k+1}^{n} b_{i,2}\right) \\ &\quad + \frac{4}{3} \sum_{k=1}^{n-1} \left(a_{k,2} \sum_{i=k+1}^{n} a_{i,2}\right) + \frac{2}{3} \left(\sum_{i=1}^{n} a_{i,2}\right) \left(\sum_{i=1}^{n} b_{i,2}\right). \end{aligned} \tag{2.3}$$

From Theorem 1.1 we have the following inequalities for the coefficients:

$$|a_{i,2}| \leq \frac{1 + 4\beta_i}{2(1 + 2\beta_i)}$$

$$|a_{i,3}| \leq \frac{(1 + 4\beta_i)(3 + 16\beta_i + 24\beta_i^2)}{12(1 + 2\beta_i)^3}$$

and

$$|b_{i,2}| \leq \frac{1 + 4\gamma_i}{2(1 + 2\gamma_i)}$$

$$|b_{i,3}| \leq \frac{(1 + 4\gamma_i)(3 + 16\gamma_i + 24\gamma_i^2)}{12(1 + 2\gamma_i)^3}$$

for $i = \overline{1, n}$. Now we will use the inequalities in (2.2) and (2.3) and we obtain:

$$\begin{aligned} |c_2| &\leq \frac{1}{2} \sum_{i=1}^{n} |b_{i,2}| + \sum_{i=1}^{n} |a_{i,2}| \\ &\leq \frac{1}{2} \left(\sum_{i=1}^{n} \frac{1 + 4\gamma_i}{2(1 + 2\gamma_i)} + \sum_{i=1}^{n} \frac{1 + 4\beta_i}{1 + 2\beta_i}\right) \end{aligned}$$

and

$$
\begin{aligned}
|c_3| &\leq \frac{1}{3}\sum_{i=1}^{n}|b_{i,3}| + \sum_{i=1}^{n}|a_{i,3}| + \frac{1}{3}\sum_{k=1}^{n-1}\left(|b_{k,2}|\sum_{i=k+1}^{n}|b_{i,2}|\right) \\
&\quad + \frac{4}{3}\sum_{k=1}^{n-1}\left(|a_{k,2}|\sum_{i=k+1}^{n}|a_{i,2}|\right) + \frac{2}{3}\left(\sum_{i=1}^{n}|a_{i,2}|\right)\left(\sum_{i=1}^{n}|b_{i,2}|\right) \\
&\leq \frac{1}{3}\left[\sum_{i=1}^{n}\frac{(1+4\gamma_i)(3+16\gamma_i+24\gamma_i^2)}{12(1+2\gamma_i)^3}\right. \\
&\qquad \left. + \sum_{k=1}^{n-1}\left(\frac{1+4\gamma_k}{2(1+2\gamma_k)}\sum_{i=k+1}^{n}\frac{1+4\gamma_i}{2(1+2\gamma_i)}\right)\right] \\
&\quad + \sum_{i=1}^{n}\frac{(1+4\beta_i)(3+16\beta_i+24\beta_i^2)}{12(1+2\beta_i)^3} \\
&\quad + \frac{2}{3}\left[2\sum_{k=1}^{n-1}\left(\frac{1+4\beta_k}{2(1+2\beta_k)}\sum_{i=k+1}^{n}\frac{1+4\beta_i}{2(1+4\beta_i)}\right)\right. \\
&\qquad \left. + \left(\sum_{i=1}^{n}\frac{1+4\beta_i}{2(1+2\beta_i)}\right)\left(\sum_{i=1}^{n}\frac{1+4\gamma_i}{2(1+2\gamma_i)}\right)\right],
\end{aligned}
$$

hence the proof is complete. □

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