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Generalization of some extremal problems on non-overlapping domains with free poles

Abstract. Some results related to extremal problems with free poles on radial systems are generalized. They are obtained by applying the known methods of geometric function theory of complex variable. Sufficiently good numerical results for γ are obtained.

1. Introduction. In geometric function theory of complex variable extremal problems on non-overlapping domains form the well-known classic direction. In the paper [1] Lavrent'ev posed and solved a problem of maximizing the product of conformal radii of two non-overlapping simply connected domains. Topics connected with the study of problems on non-overlapping domains was developed in papers [1]–[21]. This paper summarizes some results obtained in [5], [2].

Let $\mathbb{N}$, $\mathbb{R}$ be the set of natural and real numbers, respectively, $\mathbb{C}$ be the complex plane, $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the one point compactification and $\mathbb{R}^+ = (0, \infty)$.

Let $r(B, a)$ be an inner radius of a domain $B \subset \overline{\mathbb{C}}$ with respect to a point $a \in B$ (see [6], [13], [3]) and $\chi(t) = \frac{1}{2}(t + t^{-1})$.

Let $n \in \mathbb{N}$. A set of points $A_n := \left\{a_k \in \mathbb{C} : k = \overline{1, n}\right\}$ is called ***n*-radial** system if $|a_k| \in \mathbb{R}^+$, $k = \overline{1, n}$, and $0 = \arg a_1 < \arg a_2 < \ldots < \arg a_n < 2\pi$.

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Denote

$$P_k(A_n) \coloneqq \{w : \arg a_k < \arg w < \arg a_{k+1}\},$$
$$\theta_k \coloneqq \arg a_k, \qquad a_{n+1} \coloneqq a_1, \quad \theta_{n+1} \coloneqq 2\pi,$$
$$\alpha_k \coloneqq \frac{1}{\pi} \arg \frac{a_{k+1}}{a_k}, \quad \alpha_{n+1} \coloneqq \alpha_1, \quad k = \overline{1, n}.$$

This work is based on application of separating transformation developed in [4]–[6]. For specific use of this method we consider a special system of conformal mappings. By $\zeta = \pi_k(w) = -i\left(e^{-i\theta_k} w\right)^{\frac{1}{\alpha_k}}, \quad k = \overline{1,n}$ we denote unique branch of multi-valued analytic function $\pi_k(w)$ performing univalent and conformal mapping $P_k(A_n)$ onto the right half plane $\operatorname{Re}\zeta > 0$.

For an arbitrary n-radial system of points $A_n = \{a_k\}$ and $\gamma \in \mathbb{R}^+$ we assume that

$$\mathcal{L}^{(\gamma)}(A_n) \coloneqq \prod_{k=1}^{n} \left[\chi\left(\left|\frac{a_k}{a_{k+1}}\right|^{\frac{1}{2\alpha_k}}\right)\right]^{1-\frac{1}{2}\gamma\alpha_k^2} \cdot \prod_{k=1}^{n} |a_k|^{1+\frac{1}{4}\gamma(\alpha_k+\alpha_{k-1})},$$

$$\mathcal{L}^{(0)}(A_n) \coloneqq \prod_{k=1}^{n} \left[\chi\left(\left|\frac{a_k}{a_{k+1}}\right|^{\frac{1}{2\alpha_k}}\right)\right] \cdot |a_k|.$$

The class of n-radial systems of points for which $\mathcal{L}^{(\gamma)}(A_n) = 1$ ($\mathcal{L}^{(0)}(A_n) = 1$) automatically includes all systems with n different points located on the unit circle.

The main purpose of this work is to obtain exact upper estimates for the functionals

$$J_n(\gamma) = r^{\gamma}(B_0, 0) \prod_{k=1}^{n} r(B_k, a_k), \tag{1}$$

$$I_n(\gamma) = [r(B_0, 0)\, r(B_\infty, \infty)]^{\gamma} \prod_{k=1}^{n} r(B_k, a_k),$$

where $\gamma \in \mathbb{R}^+$, $A_n = \{a_k\}_{k=1}^n$ is n-radial system of points, $a_0 = 0$, and $\{B_k\}_{k=0}^n$ is a system of non-overlapping domains (i.e. $B_p \cap B_j = \emptyset$ if $p \neq j$) such that $a_k \in B_k$, $k = \overline{0, n}$.

2. Main results.

Theorem 1. *Let $n \in \mathbb{N}$, $n \geq 2$ and $\gamma \in (0, 1]$. Then for any n-radial system of points $A_n = \{a_k\}_{k=1}^n$ such that $\mathcal{L}^{(\gamma)}(A_n) = 1$ and any system of non-overlapping domains B_k, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{1, n}$, $a_0 = 0 \in B_0$ we have the inequality*

$$J_n(\gamma) \leq \frac{4^{n+\frac{\gamma}{n}} \gamma^{\frac{\gamma}{n}} n^n}{(n^2-\gamma)^{n+\frac{\gamma}{n}}} \left(\frac{n-\sqrt{\gamma}}{n+\sqrt{\gamma}}\right)^{2\sqrt{\gamma}}. \tag{2}$$

Equality in (2) *is achieved when* a_k *and* B_k, $k=\overline{0,n}$ *are, respectively, poles and circular domains of the quadratic differential*

$$Q(w)dw^2 = -\frac{(n^2-\gamma)w^n+\gamma}{w^2(w^n-1)^2}\,dw^2.$$

Theorem 2. *Let* $n\in\mathbb{N}$, $n\geq 2$ *and* $\gamma=\frac{1}{2}$. *Then for any* n*-radial system of points* $A_n=\{a_k\}_{k=1}^n$ *such that* $\mathcal{L}^{(0)}(A_n)=1$ *and any system of non-overlapping domains* B_k, B_0, B_∞, $a_k\in B_k\subset\overline{\mathbb{C}}$, $k=\overline{1,n}$, $a_0=0\in B_0\subset\overline{\mathbb{C}}$, $a_\infty=\infty\in B_\infty\subset\overline{\mathbb{C}}$ *we have the inequality*

$$(3)\qquad [r(B_0,0)\,r(B_\infty,\infty)]^{\frac{1}{2}}\prod_{k=1}^{n} r(B_k,a_k)\leq \frac{2^{2n+\frac{1}{n}}}{(n^2-2)^{\frac{1}{n}+\frac{n}{2}}}\left(\frac{n-\sqrt{2}}{n+\sqrt{2}}\right)^{\sqrt{2}}.$$

Equality in (3) *is achieved, when* a_k *and* B_k *are, respectively, poles and circular domains of the quadratic differential*

$$Q(w)dw^2 = -\frac{w^{2n}+w^n(2n^2-2)+1}{w^2(w^n-1)^2}\,dw^2.$$

Theorem 3. *Let* $\gamma\in(0;\gamma_2^0]$, $\gamma_2^0=1.1$, $\varepsilon=0.25$. *Then for any* 2*-radial system of points* $A_2=\{a_k\}_{k=1}^2$ *such that* $\mathcal{L}^{(\gamma)}(A_2)=1$, $1-\varepsilon<|a_k|<1+\varepsilon$, $k=\overline{1,2}$ *and any system of non-overlapping domains* $\{B_k\}_{k=0}^2$, $a_k\in B_k$, $k=\overline{0,2}$, $a_0=0\in B_0$, *we have the inequality*

$$r^\gamma(B_0,0)\prod_{k=1}^{2} r(B_k,a_k)\leq r^\gamma(D_0,0)\prod_{k=1}^{2} r(D_k,d_k),$$

where D_k, d_k, $k=\overline{0,2}$, $d_0=0$, *are circular domains and poles of the quadratic differential*

$$Q(w)dw^2 = -\frac{(4-\gamma)w^2+\gamma}{w^2(w^2-1)^2}\,dw^2.$$

Proof of Theorem 1. We use the method due to Bakhtin [2]–[3] and properties of separating transformation (see [4]–[7], [3], [8]). We make separating transformation of domains $\{B_k\}_{k=1}^n$. Suppose

$$P_k := P_k(A_n) := \{w\in\mathbb{C}\backslash\{0\} : \theta_k<\arg w<\theta_{k+1}\}.$$

Consider the introduced system of functions $\zeta=\pi_k(w)=-i\left(e^{-i\theta_k}w\right)^{\frac{1}{\alpha_k}}$, $k=\overline{1,n}$.

Let $\Omega_k^{(1)}$, $k=\overline{1,n}$ be a domain of the plane $\mathbb{C}_\zeta$ obtained by combining the connected component $\pi_k(B_k\bigcap\overline{P}_k)$ containing a point $\pi_k(a_k)$, with its symmetrical reflection with respect to the imaginary axis.

By $\Omega_k^{(2)}$, $k=\overline{1,n}$, we denote the domain of the plane $\mathbb{C}_\zeta$, obtained by combining the connected component $\pi_k(B_{k+1}\bigcap\overline{P}_k)$, containing the point

$\pi_k(a_{k+1})$, with its symmetrical reflection with respect to the imaginary axis, $B_{n+1} := B_1$, $\pi_n(a_{n+1}) := \pi_n(a_1)$.

Besides, by $\Omega_k^{(0)}$ we denote the domain of $\mathbb{C}_\zeta$, obtained by combining the connected component $\pi_k(B_0 \bigcap \overline{P}_k)$, containing the point $\zeta = 0$, with its symmetrical reflection with respect to the imaginary axis. Denote $\pi_k(a_k) := \omega_k^{(1)}$, $\pi_k(a_{k+1}) := \omega_k^{(2)}$, $k = \overline{1,n}$, $\pi_n(a_{n+1}) := \omega_n^{(2)}$.

The definition of π_k implies that

$$|\pi_k(w) - \omega_k^{(1)}| \sim \frac{1}{\alpha_k}|a_k|^{\frac{1}{\alpha_k}-1} \cdot |w - a_k|, \quad w \to a_k, \quad w \in \overline{P_k},$$

$$|\pi_k(w) - \omega_k^{(2)}| \sim \frac{1}{\alpha_k}|a_{k+1}|^{\frac{1}{\alpha_k}-1} \cdot |w - a_{k+1}|, \quad w \to a_{k+1}, \quad w \in \overline{P_k},$$

$$|\pi_k(w)| \sim |w|^{\frac{1}{\alpha_k}}, \quad w \to 0, \quad w \in \overline{P_k}.$$

Then, using results of papers [4]–[7], [3], we obtain the inequalities

$$r\left(B_k, a_k\right) \le \left[\frac{r\left(\Omega_k^{(1)}, \omega_k^{(1)}\right) \cdot r\left(\Omega_k^{(2)}, \omega_k^{(2)}\right)}{\frac{1}{\alpha_k}|a_k|^{\frac{1}{\alpha_k}-1} \cdot \frac{1}{\alpha_{k-1}}|a_k|^{\frac{1}{\alpha_{k-1}}-1}}\right]^{\frac{1}{2}}, \tag{4}$$

$$k = \overline{1,n}, \quad \Omega_0^{(2)} := \Omega_n^{(2)}, \quad \omega_0^{(2)} := \omega_n^{(2)},$$

$$r\left(B_0, 0\right) \le \left[\prod_{k=1}^{n} r^{\alpha_k^2}\left(\Omega_k^{(0)}, 0\right)\right]^{\frac{1}{2}}. \tag{5}$$

Repeating arguments given in [3] in the proof of Theorem 5.2.1 and taking into account introduced sets of domains $\{P_k\}_{k=1}^n$, functions $\{\pi_k\}_{k=1}^n$ and numbers $\{\theta_k\}_{k=1}^n$, we obtain the following inequality for the investigated functional (1):

$$\begin{aligned} J_n(\gamma) &\le \prod_{k=1}^{n}\left[r\left(\Omega_k^{(0)}, 0\right)\right]^{\frac{\alpha_k^2}{2}\gamma} \cdot \prod_{k=1}^{n}\left[\frac{r\left(\Omega_{k-1}^{(2)}, \omega_{k-1}^{(2)}\right) r\left(\Omega_k^{(1)}, \omega_k^{(1)}\right)}{\frac{1}{\alpha_{k-1}\cdot\alpha_k}|a_k|^{\frac{1}{\alpha_{k-1}}-1} \cdot |a_k|^{\frac{1}{\alpha_k}-1}}\right]^{\frac{1}{2}} \\ &= \prod_{k=1}^{n}\alpha_k \cdot \prod_{k=1}^{n}\frac{|a_k|}{|a_k a_{k+1}|^{\frac{1}{2\alpha_k}}} \\ &\quad \times \left[\prod_{k=1}^{n} r^{\gamma\alpha_k^2}\left(\Omega_k^{(0)}, 0\right)\prod_{k=1}^{n} r\left(\Omega_k^{(1)}, \omega_k^{(1)}\right) r\left(\Omega_k^{(2)}, \omega_k^{(2)}\right)\right]^{\frac{1}{2}}. \end{aligned} \tag{6}$$

Expression in parentheses of the last formula in (6) is a product of the functional $r^{\beta^2}\left(\Omega_k^{(0)}, 0\right) r\left(\Omega_k^{(1)}, \omega_k^{(1)}\right) r\left(\Omega_k^{(2)}, \omega_k^{(2)}\right)$ on triples of domains $\left(\Omega_k^{(0)}, \Omega_k^{(1)}, \Omega_k^{(2)}\right)$ of the plane $\mathbb{C}_\zeta$.

It is known [15] that the functional

$$Y_3(\sigma_1,\sigma_2,\sigma_3)=\frac{r^{\sigma_1}(D_1,d_1)\cdot r^{\sigma_2}(D_2,d_2)\cdot r^{\sigma_3}(D_3,d_3)}{|d_1-d_2|^{\sigma_1+\sigma_2-\sigma_3}\cdot|d_1-d_3|^{\sigma_1-\sigma_2+\sigma_3}\cdot|d_2-d_3|^{-\sigma_1+\sigma_2+\sigma_3}},$$

$\sigma_k\in\mathbb{R}^+$, $d_k\in D_k\subset\overline{\mathbb{C}}$, $D_k\cap D_p=\varnothing$, $k=1,2,3$, $p=1,2,3$, $k\neq p$, is invariant under all conformal automorphisms of the complex plane $\overline{\mathbb{C}}$.

With this relation in mind, the following estimate holds:

$$J_n(\gamma)\le\left(\prod_{k=1}^{n}\alpha_k\right)\cdot\prod_{k=1}^{n}\frac{|a_k|}{|a_ka_{k+1}|^{\frac{1}{2\alpha_k}}}$$
$$\times\left\{\prod_{k=1}^{n}\frac{r^{\gamma\alpha_k^2}\left(\Omega_k^{(0)},0\right)\cdot r\left(\Omega_k^{(1)},\omega_k^{(1)}\right)\cdot r\left(\Omega_k^{(2)},\omega_k^{(2)}\right)}{|\omega_k^{(1)}\cdot\omega_k^{(2)}|^{\gamma\alpha_k^2}|\omega_k^{(1)}-\omega_k^{(2)}|^{2-\gamma\alpha_k^2}}\right\}^{\frac12}$$
$$\times\left[\prod_{k=1}^{n}|\omega_k^{(1)}\cdot\omega_k^{(2)}|^{\gamma\alpha_k^2}|\omega_k^{(1)}-\omega_k^{(2)}|^{2-\gamma\alpha_k^2}\right]^{\frac12}.$$

Note that $|\omega_k^{(1)}|=|a_k|^{\frac{1}{\alpha_k}}$, $|\omega_k^{(2)}|=|a_{k+1}|^{\frac{1}{\alpha_k}}$, $|\omega_k^{(1)}-\omega_k^{(2)}|=|a_k|^{\frac{1}{\alpha_k}}+|a_{k+1}|^{\frac{1}{\alpha_k}}$.

Taking into account these equalities, we obtain

$$J_n(\gamma)\le\left(\prod_{k=1}^{n}\alpha_k\right)\cdot\prod_{k=1}^{n}\frac{|a_k|}{|a_ka_{k+1}|^{\frac{1}{2\alpha_k}}}$$
$$\times\left(\prod_{k=1}^{n}|\omega_k^{(1)}-\omega_k^{(2)}|\right)\left(\prod_{k=1}^{n}\frac{|\omega_k^{(1)}\cdot\omega_k^{(2)}|}{|\omega_k^{(1)}-\omega_k^{(2)}|}\right)^{\frac{\gamma\alpha_k^2}{2}}$$
$$\times\left\{\prod_{k=1}^{n}\frac{r^{\gamma\alpha_k^2}\left(\Omega_k^{(0)},0\right)\cdot r\left(\Omega_k^{(1)},\omega_k^{(1)}\right)\cdot r\left(\Omega_k^{(2)},\omega_k^{(2)}\right)}{|\omega_k^{(1)}\cdot\omega_k^{(2)}|^{\gamma\alpha_k^2}|\omega_k^{(1)}-\omega_k^{(2)}|^{2-\gamma\alpha_k^2}}\right\}^{\frac12}$$
$$=2^n\cdot\left(\prod_{k=1}^{n}\alpha_k\right)\cdot\prod_{k=1}^{n}\chi\left(\left|\frac{a_k}{a_{k+1}}\right|^{\frac{1}{2\alpha_k}}\right)|a_k|$$
$$\times 2^{-\frac{\gamma}{2}\sum\limits_{k=1}^{n}\alpha_k}\left[\prod_{k=1}^{n}\chi\left(\left|\frac{a_k}{a_{k+1}}\right|^{\frac{1}{2\alpha_k}}\right)\right]^{-\frac{\gamma\alpha_k^2}{2}}\left(\prod_{k=1}^{n}\left|\frac{a_{k+1}}{a_k}\right|\right)^{\frac{\gamma\alpha_k^2}{2}}$$
$$\times\left\{\prod_{k=1}^{n}\frac{r^{\gamma\alpha_k^2}\left(\Omega_k^{(0)},0\right)\cdot r\left(\Omega_k^{(1)},\omega_k^{(1)}\right)\cdot r\left(\Omega_k^{(2)},\omega_k^{(2)}\right)}{|\omega_k^{(1)}\cdot\omega_k^{(2)}|^{\gamma\alpha_k^2}|\omega_k^{(1)}-\omega_k^{(2)}|^{2-\gamma\alpha_k^2}}\right\}^{\frac12}$$

$$= 2^{n-\frac{\gamma}{2}\sum\limits_{k=1}^{n}\alpha_k^2} \cdot \left(\prod_{k=1}^{n} \alpha_k\right) \cdot \prod_{k=1}^{n} \left[\chi\left(\left|\frac{a_k}{a_{k+1}}\right|^{\frac{1}{2\alpha_k}}\right)\right]^{1-\frac{\gamma\alpha_k^2}{2}}$$

$$\times \prod_{k=1}^{n} |a_k|^{1+\frac{1}{4}\gamma(\alpha_k+\alpha_{k-1})}$$

$$\times \left\{\prod_{k=1}^{n} \frac{r^{\gamma\alpha_k^2}\left(\Omega_k^{(0)}, 0\right) \cdot r\left(\Omega_k^{(1)}, \omega_k^{(1)}\right) \cdot r\left(\Omega_k^{(2)}, \omega_k^{(2)}\right)}{|\omega_k^{(1)} \cdot \omega_k^{(2)}|^{\gamma\alpha_k^2} |\omega_k^{(1)} - \omega_k^{(2)}|^{2-\gamma\alpha_k^2}}\right\}^{\frac{1}{2}}$$

$$= 2^{n-\frac{\gamma}{2}\sum\limits_{k=1}^{n}\alpha_k^2} \cdot \left(\prod_{k=1}^{n} \alpha_k\right) \cdot \mathcal{L}^{(\gamma)}\left(A_n\right)$$

$$\times \left\{\prod_{k=1}^{n} \frac{r^{\gamma\alpha_k^2}\left(\Omega_k^{(0)}, 0\right) \cdot r\left(\Omega_k^{(1)}, \omega_k^{(1)}\right) \cdot r\left(\Omega_k^{(2)}, \omega_k^{(2)}\right)}{|\omega_k^{(1)} \cdot \omega_k^{(2)}|^{\gamma\alpha_k^2} |\omega_k^{(1)} - \omega_k^{(2)}|^{2-\gamma\alpha_k^2}}\right\}^{\frac{1}{2}}.$$

For each $k = \overline{1,n}$ we can easily define conformal automorphism $\zeta = T_k(z)$ of complex numbers of the plane $\overline{\mathbb{C}}$ such that $T_k(0) = 0,\ T_k\left(\omega_k^{(s)}\right) = (-1)^s \cdot i$, $G_k^{(q)} := T_k\left(\Omega_k^{(q)}\right),\ k = \overline{1,n},\ s = 1,2,\ q = 0,1,2.$

Then

$$\left\{\prod_{k=1}^{n} \frac{r^{\gamma\alpha_k^2}\left(\Omega_k^{(0)}, 0\right) \cdot r\left(\Omega_k^{(1)}, \omega_k^{(1)}\right) \cdot r\left(\Omega_k^{(2)}, \omega_k^{(2)}\right)}{|\omega_k^{(1)} \cdot \omega_k^{(2)}|^{\gamma\alpha_k^2} |\omega_k^{(1)} - \omega_k^{(2)}|^{2-\gamma\alpha_k^2}}\right\}^{\frac{1}{2}}$$

$$= \left\{\prod_{k=1}^{n} \frac{r^{\alpha_k^2\gamma}\left(G_k^{(0)}, 0\right) \cdot r\left(G_k^{(1)}, -i\right) \cdot r\left(G_k^{(2)}, i\right)}{2^{2-\gamma\alpha_k^2}}\right\}^{\frac{1}{2}}.$$

Then using results of [3], [15], we obtain

$$J_n(\gamma) \leq 2^{n-\frac{\gamma}{2}\sum\limits_{k=1}^{n}\alpha_k^2} \cdot \left(\prod_{k=1}^{n} \alpha_k\right) \cdot \mathcal{L}^{(\gamma)}\left(A_n\right)$$

$$\times \prod_{k=1}^{n} \left\{\frac{r^{\alpha_k^2\gamma}\left(G_k^{(0)}, 0\right) \cdot r\left(G_k^{(1)}, -i\right) \cdot r\left(G_k^{(2)}, i\right)}{2^{2-\gamma\alpha_k^2}}\right\}^{\frac{1}{2}}$$

$$= 2^{n-\frac{\gamma}{2}\sum\limits_{k=1}^{n}\alpha_k^2} \left(\prod_{k=1}^{n} \alpha_k\right) \cdot \mathcal{L}^{(\gamma)}(A_n) \cdot 2^{-n+\frac{\gamma}{2}\sum\limits_{k=1}^{n}\alpha_k^2}$$

$$\times \left[\prod_{k=1}^{n} r^{\alpha_k^2 \gamma} \left(G_k^{(0)}, 0 \right) \cdot r \left(G_k^{(1)}, -i \right) \cdot r \left(G_k^{(2)}, i \right) \right]^{\frac{1}{2}}$$

$$\leq \left(\prod_{k=1}^{n} \alpha_k \right) \cdot \mathcal{L}^{(\gamma)}(A_n) \cdot \left[\prod_{k=1}^{n} r^{\alpha_k^2 \gamma} \left(G_k^{(0)}, 0 \right) \cdot r \left(G_k^{(1)}, -i \right) \cdot r \left(G_k^{(2)}, i \right) \right]^{\frac{1}{2}}.$$

Following the paper [15], we have

$$J_n(\gamma) \leq \gamma^{-\frac{n}{2}} \left[\prod_{k=1}^{n} \Psi\left(\beta_k\right) \right]^{1/2}, \tag{7}$$

where $\Psi(\beta) = 2^{\beta^2+6} \cdot \beta^{\beta^2+2}(2-\beta)^{-\frac{1}{2}(2-\beta)^2} \cdot (2+\beta)^{-\frac{1}{2}(2+\beta)^2}$, $\beta \in [0, 2]$.

Similarly to [5], we consider the next extremal problem:

$$\prod_{k=1}^{n} \Psi(\beta_k) \to \sup; \quad \sum_{k=1}^{n} \beta_k = 2\sqrt{\gamma},\ \beta_k = \alpha_k \sqrt{\gamma},\ 0 < \beta_k \leq 2.$$

Necessary conditions have the form

$$\frac{\Psi'(\beta_k)}{\Psi(\beta_k)} = \frac{-\lambda}{\prod\limits_{k=1}^{n} \Psi(\beta_k)}, \quad k = \overline{1, n}.$$

We will show that all β_k are equal. We investigate behavior of the function $F(\beta) = \frac{\Psi'(\beta)}{\Psi(\beta)} = 2\beta \ln(2\beta) + \frac{2}{\beta} + (2-\beta)\ln(2-\beta) - (2+\beta)\ln(2+\beta)$ on interval $\beta \in [0, 2]$. It is strictly decreasing on the interval $(0; \beta_0]$, $\beta_0 \in (1.32; 1.33)$ and increasing on $[\beta_0; 2)$. Then we use the method of the proof of Theorem 4 [5] and obtain that unique solution of the extremal problem is the point $(\frac{2}{n}, \ldots, \frac{2}{n})$. Estimates (4), (5), (7) yield inequality (2) of Theorem 1. The case of equality is verified directly and Theorem 1 is proved. □

Dubinin proved Theorem 1 if $\gamma = 1$ and for any distinct points a_k that lie on the unit circle and any non-overlapping domains B_k (see [5], [8]).

Proof of Theorem 2. We retain all notation for separating transformation of domains introduced in the proof of Theorem 1 for domains B_k, $k = \overline{0, n}$. By $\Omega_k^{(\infty)}$ we denote the domain of plane $\mathbb{C}_\zeta$, obtained by combining the connected component $\pi_k(B_\infty \bigcap \overline{E}_k)$ containing the point $\zeta = \infty$ with its symmetrical reflection with respect to the imaginary axis. The family $\left\{ \Omega_k^{(\infty)} \right\}_{k=1}^{n}$ is a result of separating transformation of an arbitrary domain B_∞, $\infty \in B_\infty \subset \overline{\mathbb{C}}$ with respect to the families $\{P_k\}_{k=1}^n$ and $\{\pi_k\}_{k=1}^n$ at the point $\zeta = \infty$.

By Theorem 2 in [5] we have

$$r(B_\infty,\infty)\le\left[\prod_{k=1}^{n} r^{\alpha_k^2}\left(\Omega_k^{(\infty)},\infty\right)\right]^{\frac12}. \tag{8}$$

Using (4), (5), (8), we obtain

$$\left[r\left(B_0,0\right)r\left(B_\infty,\infty\right)\right]^{\frac12}\prod_{k=1}^{n} r\left(B_k,a_k\right)\le 2^n\cdot\left(\prod_{k=1}^{n}\alpha_k\right)\cdot\mathcal{L}^{(0)}(A_n)$$

$$\times\left[\prod_{k=1}^{n} r^{\alpha_k^2\gamma}\left(\Omega_k^{(0)},0\right)\cdot r^{\alpha_k^2\gamma}\left(\Omega_k^{(\infty)},\infty\right)\cdot\frac{r\left(\Omega_k^{(1)},\omega_k^{(1)}\right)\cdot r\left(\Omega_k^{(2)},\omega_k^{(2)}\right)}{\left|\omega_k^{(1)}-\omega_k^{(2)}\right|^2}\right]^{\frac12}.$$

Theorem 6 in [5] gives

$$\begin{aligned}&\left[r\left(B_0,0\right)r\left(B_\infty,\infty\right)\right]^{\alpha_k^2\gamma}\cdot\frac{r\left(B_1,a_1\right)r\left(B_2,a_2\right)}{|a_1-a_2|^2}\le\Psi(\beta)\\&\quad=\beta^{2\beta^2}\cdot|1-\beta|^{-(1-\beta)^2}\cdot(1+\beta)^{-(1+\beta)^2},\quad 0<\beta\le\sqrt2.\end{aligned} \tag{9}$$

Inequality (9) was obtained by Dubinin using the results of Kolbina [15]. Similarly to [5], we consider the extremal problem:

$$\prod_{k=1}^{n}\Psi(\beta_k)\to\sup;\quad \sum_{k=1}^{n}\beta_k=\sqrt2.$$

We introduce a function $F(\beta)=\frac{\Psi'(\beta)}{\Psi(\beta)}$. Calculations show that this function is decreasing on the interval $(0;\beta_0]$ and increases on $[\beta_0;\sqrt2)$, $0.85<\beta_0<1$. Further, as in the proof of Theorem 6 in [5], we verify that the unique solution of the extremal problem is the point $(\frac{\sqrt2}{n},\dots,\frac{\sqrt2}{n})$. Estimates (4), (5), (8), (9) yield the inequality (3) of Theorem 2. The case of equality is verified directly. Theorem 2 is proved. □

Proof of Theorem 3. The proof is based on application of separating transformation, developed in details in [6]. According to the conditions of Theorem 3, $a_0=0$, $1-\varepsilon<|a_k|<1+\varepsilon$, $k=\overline{1,2}$. Assume

$$0=\arg a_1<\arg a_2<2\pi.$$

Let $\alpha_1:=\frac1\pi\cdot(\arg a_2-\arg a_1)$, $\alpha_2:=\frac1\pi\cdot(2\pi-\arg a_2)$, $P_k:=\{w:\arg a_k<\arg w<\arg a_{k+1}\}$, $k=\overline{1,2}$, $\arg a_3=2\pi$, $P_0:=P_2$, $P_3:=P_1$.

The family of two symmetrical domains $\{D_k^{(1)};D_{k-1}^{(2)}\}$ with respect to the imaginary axis is called a result of separating transformation of the domain B_k.

Further, as in Theorem 5.2.1 in [3], using the separating transformation we obtain

$$(10)\qquad J_2(\gamma) \le \mathcal{L}^{(\gamma)}(A_2)\left[\prod_{k=1}^{2} \alpha_k^2 r^{\gamma\cdot\alpha_k^2}(D_0,0)\cdot r(D_1,1)\cdot r(D_2,-1)\right]^{\frac{1}{2}}.$$

We will prove that for $\alpha_0 \ge \frac{2}{\sqrt{\gamma}}$, $\alpha_0 = \max\{\alpha_1,\alpha_2\}$ extremal configurations different than those referred in Theorem 3 do not exist. For this we find a value of the functional (see Theorem 5.2.3 in [3])

$$J_2^0(\gamma) = r^{\gamma}(D_0,0)\cdot\prod_{k=1}^{2} r(D_k,d_k) = 4\cdot\frac{(\gamma)^{\frac{\gamma}{2}}}{\left(1-\frac{\gamma}{4}\right)^{2+\frac{\gamma}{2}}}\cdot\left(\frac{1-\frac{\sqrt{\gamma}}{2}}{1+\frac{\sqrt{\gamma}}{2}}\right)^{2\sqrt{\gamma}}$$

if $\gamma = 1.1$. We have that $J_2^0(1.1) \approx 0.8315$.

Denote $r(B_0,0) = r_0$, $r(B_1,a_1) = r_1$, $r(B_2,a_2) = r_2$. The Lavrent'ev's theorem [17] gives $r_0r_1 < |a_1|^2$, $r_0r_2 < |a_2|^2$, $r_0^2r_1r_2 < |a_1|^2|a_2|^2 \Longrightarrow r_1r_2 < \frac{|a_1|^2|a_1|^2}{r_0^2}$.

Then $r^{\gamma}(B_0,a_0)\cdot\prod\limits_{k=1}^{2} r(B_k,a_k) = r_0^{\gamma}\cdot\prod\limits_{k=1}^{2} r(B_k,a_k) \le r_0^{\gamma}\cdot\frac{|a_1|^2\cdot|a_2|^2}{r_0^2} \le J_2^0(\gamma) \Rightarrow r_0 \ge \left(\frac{|a_1|^2\cdot|a_2|^2}{J_2^0(\gamma)}\right)^{\frac{1}{2-\gamma}}$. If $r_0 \ge \left(\frac{|a_1|^2\cdot|a_2|^2}{J_2^0(\gamma)}\right)^{\frac{1}{2-\gamma}}$, then the extremal configurations do not exist. Consider the case $r_0 \le \left(\frac{|a_1|^2\cdot|a_2|^2}{J_2^0(\gamma)}\right)^{\frac{1}{2-\gamma}}$.

$$J_2(\gamma) \le r_0^{\gamma}|a_1-a_2|^2 = r_0^{\gamma}\left((|a_1|-|a_2|)^2 + 4|a_1|\cdot|a_2|\sin^2(2-\alpha_0)\frac{\pi}{2}\right)$$
$$\le \left(\frac{|a_1|\cdot|a_2|}{J_2^0(\gamma)}\right)^{\frac{\gamma}{2-\gamma}}\cdot\left((|a_1|-|a_2|)^2 + 4|a_1|\cdot|a_2|\sin^2(2-\alpha_0)\frac{\pi}{2}\right) \le J_2^0(1.1).$$

Substituting $\varepsilon = 0.25$, $\gamma = 1.1$, $n = 2$, $|a_1| = 1-\varepsilon$, $|a_2| = 1+\varepsilon$, $J_2^0(1.1) = 0.8315$ in the last inequality, we obtain

$$(11)\quad (1+\varepsilon)^{\frac{1.1}{2-1,1}}\left(4\varepsilon^2 + 4(1+\varepsilon)\sin^2\left(2-\frac{2}{\sqrt{1.1}}\right)\frac{\pi}{2}\right) \le J_2^0(1.1)^{1+\frac{1.1}{2-1.1}}.$$

Performing calculations of right and left sides of the inequality (11), we have $0.6085 < 0.6663$. From this it follows that if $\varepsilon = 0.25$, then inequality (11) is true. Hence $J = \frac{J_2(1.1)}{J_2^0(1.1)} \le \frac{0.7316}{0.8315} = 0.8798 < 1$, i.e. if $\alpha_0 > \frac{2}{\sqrt{\gamma}}$, $J_2(\gamma) < J_2^0(\gamma)$ then the maximum value of the functional $J_2(\gamma)$ for such domains is not attained. Then $\alpha_0 \le \frac{2}{\sqrt{\gamma}}$ and we can apply inequality (10).

Using a result obtained in the proof of Theorem 4 in [5], we can write the following inequality

$$J_2(\gamma) \le \frac{1}{\sqrt{\gamma}}\cdot\left[\prod_{k=1}^{2} 2^{\sigma_k^2+6}\cdot\sigma_k^{\sigma_k^2+2}\cdot(2-\sigma_k)^{-\frac{1}{2}(2-\sigma_k)^2}(2+\sigma_k)^{-\frac{1}{2}(2+\sigma_k)^2}\right]^{\frac{1}{2}},$$

where $\sigma_k = \sqrt{\gamma} \cdot \alpha_k$. Consider the function

$$\Psi(\sigma) = 2^{\sigma^2+6} \cdot \sigma^{\sigma^2+2} \cdot (2-\sigma)^{-\frac{1}{2}(2-\sigma)^2}(2+\sigma)^{-\frac{1}{2}(2+\sigma)^2},$$

$\sigma \in [0, 2]$ and we will conduct detailed investigation of its graph on this interval (see Fig.1).

$\Psi(\sigma)$ is logarithmically convex on interval $[0; x_0]$, $x_0 \approx 1.32$. On $[0; x_1]$, $x_1 \approx 1.05$ the function increases from $\Psi(0) = 0$ to $\Psi(x_1) \approx 0.9115$, and it decreases on interval $[x_1; x_2]$, $x_2 \approx 1.6049$ to $\Psi(x_2) \approx 0.86$, and on $[x_2; 2]$ it increases to $\Psi(2) = 1$. $x_3 \approx 1.9$, $\Psi(x_3) = \Psi(x_1) \approx 0.9115$.

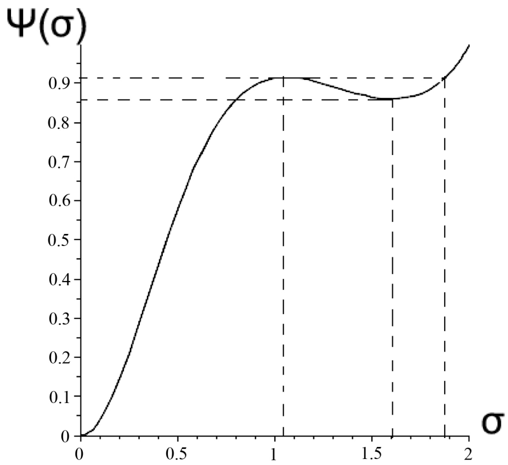

Figure 1.

Using equality $\sigma_1 + \sigma_2 = 2\sqrt{\gamma}$, we will prove that $\Psi(\sigma_1) \cdot \Psi(\sigma_2) \leq (\Psi(x_1))^2 \approx 0.8308$. For $\sigma \in [0; x_0]$ we make appropriate conclusion from the logarithmic convexity of the function $\Psi(\sigma)$. For $\sigma_2 \in [x_0; x_3]$ from properties of the graph of the function $\Psi(\sigma)$, we have $\Psi(\sigma_2) \leq \Psi(x_1)$ and $\Psi(\sigma_1) \leq \Psi(x_1)$ and thus $\Psi(\sigma_1) \cdot \Psi(\sigma_2) \leq (\Psi(x_1))^2$.

If $\sigma_2 \in [x_3; 2]$ then $\Psi(\sigma_2) < \Psi(2) = 1$, $\Psi(\sigma_1) < \Psi(0, 2) \ll 0.4$, and hence $\Psi(\sigma_1) \cdot \Psi(\sigma_2) < 0, 4 < (\Psi(x_1))^2$. So, $J_2(\gamma) \leq J_2^0(\gamma)$. Inequality (11) is true and Theorem 3 is proved. □

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