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The Ramsey numbers for some subgraphs of generalized wheels versus cycles and paths

Abstract. The Ramsey number $R(G, H)$ for a pair of graphs G and H is defined as the smallest integer n such that, for any graph F on n vertices, either F contains G or $\overline{F}$ contains H as a subgraph, where $\overline{F}$ denotes the complement of F. We study Ramsey numbers for some subgraphs of generalized wheels versus cycles and paths and determine these numbers for some cases. We extend many known results studied in [5, 14, 18, 19, 20]. In particular we count the numbers $R(K_1 + L_n, P_m)$ and $R(K_1 + L_n, C_m)$ for some integers m, n, where L_n is a linear forest of order n with at least one edge.

1. Introduction. We consider a simple graph $G = (V(G), E(G))$. Let P_i denote a path consisting of i vertices and let kP_i denote k disjoint copies of P_i. By C_m we denote a cycle of order m. For two vertex disjoint graphs G and F by $G \cup F$ we denote the vertex disjoint union of G and F. By $\overline{G}$ we denote the complement of the graph G.

The graph $K_1 + mK_2$ is called a fan, denoted by F_m. For integer $m \geq 3$ the graph $K_1 + C_m$ is called a wheel, and denoted by $W_{1,m}$ or equivalently by W_m, where the single vertex of K_1 is called the hub and all vertices of C_m are called the rims of the wheel. Moreover, for integer $t \geq 1$ and $m \geq 3$ we define a generalized wheel $W_{t,m}$ as $K_t + C_m$. Let L_n be a linear forest of order n with at least one edge.

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The Ramsey number $R(G, H)$ for a pair of graphs G and H is defined as the smallest integer n such that, for any graph F on n vertices, either F contains G or $\overline{F}$ contains H as a subgraph, where $\overline{F}$ denotes the complement of F.

The chromatic number $\chi(G)$ of a graph G is the smallest number of colours needed to colour the vertices of G so that no two adjacent vertices have the same colour.

A connected graph H is G-good if $R(G, H) = (\chi(G) - 1)(|V(H)| - 1) + s(G)$, where $s(G)$ is the surplus of H defined as the minimum cardinality of colour classes over all chromatic colourings of $V(G)$.

Recently many results have been obtained for Ramsey numbers of cycles versus fans and wheels. For instance Burr and Erdős [2] showed that $R(C_3, W_n) = 2n+1$ for $n \geq 5$, Radziszowski and Xia [11] gave a method for counting the Ramsey numbers $R(C_3, G)$, where G is either a path, a cycle or a wheel. Surahmat et al. [15, 16, 17] showed that $R(C_n, W_m) = 2n - 1$ for even m and $n \geq 5m/2 - 1$ and $R(C_n, W_m) = 3n - 2$ for odd m and $n > (5m - 9)/2$. Zhang et al. [19] determined $R(C_n, W_m) = 3n - 2$ for m odd, $n \geq m$ and $n \leq 20$. More recent results are presented later in Theorems 2.7, 2.8, 2.9, 2.17, 2.18 and 2.19.

The aim of this paper is to improve some results by reducing the lower bound for n. Also we will establish Ramsey numbers for some new graphs versus paths or cycles.

2. Theorems. The following lower bound on Ramsey numbers is well known in graph Ramsey theory.

Theorem 2.1 (Burr [1]). *Let G be a connected graph and H be a graph with $|V(G)| \geq s(H)$, where $s(H)$ is the surplus of H. Let $\chi(H)$ be the chromatic number of H.*

Then $R(G, H) \geq (|V(G)| - 1)(\chi(H) - 1) + s(H)$.

Theorem 2.2 (Faudree et al. [6]).

$$R(C_m, P_n) = \begin{cases} 2n - 1, & 3 \leq m \leq n,\ m \text{ odd}, \\ n - 1 + \frac{m}{2}, & 4 \leq m \leq n,\ m \text{ even}, \\ \max\{m - 1 + \lfloor \frac{n}{2} \rfloor, 2n - 1\}, & 2 \leq n \leq m,\ m \text{ odd}, \\ m - 1 + \lfloor \frac{n}{2} \rfloor, & 2 \leq n \leq m,\ m \text{ even}. \end{cases}$$

Theorem 2.3 (Lin et al. [9]). *Let a tree T_n be G-good graph, where $s(G) = 1$. Then T_n is $(K_1 + G)$-good graph.*

Note that by the first line of Theorem 2.2 we get that P_n is C_m-good for odd m, where $3 \leq m \leq n$.

Considering the third line, we have $2n - 1 \geq m - 1 + \frac{n-1}{2}$ with $m \geq n \geq \frac{2}{3}m - \frac{1}{3}$ for n odd and we have $2n - 1 \geq m - 1 + \frac{n}{2}$ with $m \geq n \geq \frac{2}{3}m$ for n even. So P_n is C_m-good for odd m, where $3 \leq m \leq \lceil \frac{3n}{2} \rceil$. By Theorem 2.2 we can see the following property.

Corollary 2.4. *Let m be odd integer and $3 \leq m \leq \lceil \frac{3n}{2} \rceil$. Then P_n is C_m-good, and $R(P_n, C_m) = 2n - 1$.*

By Corollary 2.4 and Theorem 2.3 we have immediately the next theorem.

Theorem 2.5. *Let m be odd integer and $3 \leq m \leq \lceil \frac{3n}{2} \rceil$. Then $R(P_n, W_{1,m}) = 3n - 2$.*

Similarly, by iterative application of Theorem 2.3, we get the result for paths versus generalized wheels as presented below.

Theorem 2.6. *Let $t \geq 1$ and let m be odd integer and $3 \leq m \leq \lceil \frac{3n}{2} \rceil$. Then $R(P_n, W_{t,m}) = (t + 2)(n - 1) + 1$.*

Similarly, we can use the following theorems proved by Chen et al. [5], Surahmat et al. [13] and Zhang [18]. First, we present the result, where m is an even integer.

Theorem 2.7 (Chen et al. [5]). *Let m be an even integer and $n \geq m-1 \geq 3$. Then $R(P_n, W_{1,m}) = 2n - 1$.*

Salman and Broersma obtained $R(P_n, W_{1,m})$ for m odd.

Theorem 2.8 (Salman and Broersma [13]). *Let $n \geq 4$ be an integer and let $m \geq 3$ be an odd integer with $3 \leq m \leq 2n-1$. Then $R(P_n, W_{1,m}) = 3n - 2$.*

Zhang expanded the results to the following.

Theorem 2.9 (Zhang [18]). *Let $n \geq 4$ be an integer and let m be an odd integer with $n + 2 \leq m \leq 2n$. Then $R(P_n, W_{1,m}) = 3n - 2$.*

Note that $\chi(K_1 + W_{1,m}) = \chi(K_2 + C_m) = 5$. So by Theorems 2.8, 2.9 and 2.3 we get the following results.

Theorem 2.10. *Let m be an odd integer where $3 \leq m \leq 2n$. Then $R(P_n, K_2 + C_m) = R(P_n, K_1 + W_{1,m}) = 4n - 3$.*

Similarly, for paths and more generalized wheels we have the following theorem.

Theorem 2.11. *Let $t \geq 1$ be an integer and let m be an odd integer, where $3 \leq m \leq 2n$. Then*

$$R(P_n, K_t + C_m) = R(P_n, W_{t,m}) = (t + 2)(n - 1) + 1.$$

Moreover, the following result is known.

Theorem 2.12 (see Radziszowski [10]). *$R(P_n, K_2 + C_m) = 3n - 2$ for m even and $n \geq m - 2$.*

Thus by Theorem 2.3 we generalize Theorem 2.12 as follows.

Theorem 2.13. *Let $t \geq 2$. Then $R(P_n, K_t + C_m) = (t + 1)(n - 1) + 1$ for m even and $n \geq m - 2$.*

Now we present Ramsey numbers for paths P_m versus $K_1 + L_n$. Note that $K_1 + L_n$ is a subgraph of $W_n = W_{1,n}$.

Theorem 2.14. *Let n, m be integers and let $m \geq n-1 \geq 3$ for n even and $m \geq n \geq 3$ for n odd. Then $R(P_m, K_1 + L_n) = 2m-1$.*

Proof. Note that $s(P_m) = \lfloor \frac{m}{2} \rfloor$ and $\chi(P_m) = 2$. So if $n+1 \geq s(P_m) = \lfloor \frac{m}{2} \rfloor$ by Theorem 2.1 with $H = P_m$ and $G = K_1 + L_n$ we get:

$$R(P_m, K_1 + L_n) \geq (\chi(P_m) - 1)(|V(K_1 + L_n)| - 1) + s(P_m) = n + \lfloor \frac{m}{2} \rfloor$$

and we have the lower bound

$$R(P_m, K_1 + L_n) \geq \begin{cases} n + \lfloor \frac{m}{2} \rfloor, & m \text{ odd}, \\ n + \frac{m}{2}, & m \text{ even} \end{cases}$$

in this case.

Note that $s(K_1 + L_n) = 1$ and $\chi(K_1 + L_n) = 3$. So by Theorem 2.1 with $H = K_1 + L_n$ and $G = P_m$ we get

$$R(P_m, K_1 + L_n) \geq (\chi(K_1 + L_n) - 1)(|V(P_m)| - 1) + s(K_1 + L_n) = 2m - 1.$$

Note that $n + \lfloor \frac{m}{2} \rfloor > 2m-1$, when $n + \frac{m-1}{2} > 2m-1$ for odd m and $n + \frac{m}{2} > 2m-1$ for even m. So it holds for $m < \frac{2}{3}n + \frac{1}{3}$ with m odd, and for $m < \frac{2}{3}n + \frac{2}{3}$ with m even.

So we can write

$$R(P_m, K_1 + L_n) \geq \max\left\{n + \lfloor \frac{m}{2} \rfloor, 2m-1\right\}$$

and

$$R(P_m, K_1 + L_n) \geq \begin{cases} n + \lfloor \frac{m}{2} \rfloor, & m < \frac{2}{3}n + \frac{1}{3},\ m \text{ odd}, \\ 2m-1, & m \geq \frac{2}{3}n + \frac{1}{3},\ m \text{ odd}, \\ n + \frac{m}{2}, & m < \frac{2}{3}n + \frac{2}{3},\ m \text{ even}, \\ 2m-1, & m \geq \frac{2}{3}m + \frac{2}{3},\ m \text{ even}. \end{cases}$$

Now the upper bound we obtain by the consideration given below. First we can see that $K_1 + L_n$ is a subgraph of $W_{1,n}$, so $R(P_m, K_1 + L_n) \leq R(W_{1,n}, P_m)$, for n even.

Then we note that $R(P_m, K_1 + L_n) \leq 2m-1$ for $m \geq n-1 \geq 3$ and n even. For n odd we can see that $K_1 + L_n$ is a subgraph of $K_1 + C_{n+1} = W_{1,n+1}$ so we know that $R(P_m, K_1 + L_n) \leq 2m-1$ for $m \geq n+1-1 \geq 3$, so $m \geq n \geq 3$. □

The following result is contained in [12] and [7], and a new simpler proof of it in [8]:

Theorem 2.15. *Let m, n be integers and $n \geq m \geq 3$.*

$$R(C_m, C_n) = \begin{cases} 2n-1, & m \text{ odd and } (m,n) \neq (3,3), \\ n - 1 + \frac{m}{2}, & m \text{ and } n \text{ even and } (m,n) \neq (4,4), \\ \max\{n - 1 + \frac{m}{2}, 2m-1\}, & m \text{ even and } n \text{ odd}. \end{cases}$$

Recently Shi obtained the Ramsey numbers of fans versus cycles.

Theorem 2.16 (Shi [14]). *$R(C_n, F_m) = 2n - 1$ for $n > 3m$.*

For Ramsey numbers of cycles versus wheels obtained in turn the following results.

Theorem 2.17 (Chen et al. [3]). *$R(C_m, W_{1,n}) = 3m - 2$ for odd $n \geq 3$ with $m \geq n$, $m \neq 3$.*

Theorem 2.18 (Chen et al. [4], Shi [14]). *$R(C_m, W_{1,n}) = 2m - 1$ for even $n \geq 4$ and $2m \geq 3n + 2$.*

Theorem 2.19 (Zhang et al. [20]). *$R(C_m, W_{1,n}) = 2n + 1$ for m odd, $n \geq 3(m-1)/2$ and $(m, n) \neq (3, 3), (3, 4)$.*

Now we present the Ramsey number for $K_1 + L_n$ versus a cycle C_m of order m for some integers m and n.

Theorem 2.20. *Let $m \geq 3$ be an integer.*

$$R(C_m, K_1 + L_n) = 2m - 1 \text{ for } m \geq \begin{cases} \frac{3}{2}n + 1, & n \text{ even}, \\ \frac{3}{2}n + \frac{5}{2}, & n \text{ odd}. \end{cases}$$

Moreover, $R(C_m, K_1 + L_n) = 2n + 1$ for m odd, $m \leq \frac{2n}{3} + 1$, $(m, n) \neq (3, 3), (3, 4)$.

Proof. By Theorem 2.1 for the case $H = C_m$ and $G = K_1 + L_n$ we get the following lower bounds.

For m odd and $s(C_m) = 1$ we have

$$R(C_m, K_1 + L_n) \geq (\chi(C_m) - 1)(|V(K_1 + L_n)| - 1) + s(C_m) = 2n + 1.$$

For m even if $n + 1 \geq \frac{m}{2}$ we have

$$R(C_m, K_1 + L_n) \geq (\chi(C_m) - 1)(|V(K_1 + L_n)| - 1) + s(C_m) = n + \frac{m}{2}.$$

So

$$R(C_m, K_1 + L_n) \geq \begin{cases} 2n + 1, & m \text{ odd}, \\ n + \frac{m}{2}, & m \text{ even and } n + 1 \geq \frac{m}{2}. \end{cases}$$

By Theorem 2.1 with $H = K_1 + L_n$ and $G = C_m$ we count the lower bound. Recall that $s(K_1 + L_n) = 1$ and $\chi(K_1 + L_n) = 3$. Thus

$$\begin{aligned} R(C_m, K_1 + L_n) &\geq (\chi(K_1 + L_n) - 1)(|V(C_m)| - 1) + s(K_1 + L_n) \\ &= 2m - 1. \end{aligned}$$

So

$$R(C_m, K_1 + L_n) \geq \begin{cases} \max\{2m - 1, 2n + 1\}, & m \text{ odd}, \\ \max\{n + \frac{m}{2}, 2m - 1\}, & m \text{ even and } n + 1 \geq \frac{m}{2}. \end{cases}$$

Finally, by two above cases we get the following lower bounds

$$R(C_m, K_1 + L_n) \geq \begin{cases} 2n+1, & m < n+1,\ m \text{ odd}, \\ 2m-1, & m \geq n+1,\ m \text{ odd}, \\ n + \frac{m}{2}, & m < \frac{2n}{3} + \frac{2}{3},\ m \text{ even}, \\ 2m-1, & m \geq \frac{2n}{3} + \frac{2}{3},\ m \text{ even}. \end{cases}$$

Thus we get the lower bound. Now the upper bound we obtain by the consideration given below.

We can see that $K_1 + L_n$ is subgraph of $W_{1,n}$, so $R(C_m, K_1 + L_n) \leq R(C_m, W_{1,n})$, n even.

By Theorem 2.18, we know that $R(C_m, K_1 + L_n) \leq 2m - 1$ for $m \geq \frac{3}{2}n + 1$ and n even.

For n odd we can see that $K_1 + L_n$ is subgraph of $W_{1,n+1}$ so we know that $R(C_m, K_1 + L_n) \leq 2m - 1$ for $m \geq \frac{3}{2}n + \frac{5}{2}$.

Now consider m odd for $m \leq \frac{2n}{3} + 1$ with $(m, n) \neq (3, 3), (3, 4)$. By Theorem 2.19 we get $R(K_1 + L_n, C_m) \leq 2n + 1$. By Theorem 2.1 we have that $R(K_1 + L_n, C_m) \geq 2n + 1$ for $m \leq n + 1$. Thus $R(K_1 + L_n, C_m) = 2n + 1$ for odd m, $m \leq \frac{2n}{3} + 1$, $(m, n) \neq (3, 3), (3, 4)$. □

Now we present the Ramsey numbers for some generalized fans $K_1 + kP_3$ versus a cycle. The graph is a special case of $K_1 + L_n$. Thus by Theorem 2.20 we get some generalization of Shi's result (see Theorem 2.16).

Corollary 2.21.

$$R(C_m, K_1 + kP_3) = 2m - 1 \text{ for } m \geq \begin{cases} \frac{9}{2}k + 1, & k \text{ even}, \\ \frac{9}{2}k + \frac{5}{2}, & k \text{ odd}. \end{cases}$$

Moreover, $R(C_m, K_1 + kP_3) = 6k + 1$ *for* m *odd,* $m \leq 2k + 1$, $(m, k) \neq (3, 1)$.

Open problem. Let

$$\varepsilon = \begin{cases} 1, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}$$

One can study $R(C_m, K_1 + L_n)$ for even m, $m \leq \lfloor (3n+1)/2 \rfloor + \varepsilon$ and odd m, $\frac{2n}{3} + 1 < m \leq \lfloor (3n+1)/2 \rfloor + \varepsilon$.

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