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The existence of solutions to the nonhomogeneous degenerate nonlinear elliptic equations

ABSTRACT. In this paper we are interested in the existence and uniqueness of solutions for Dirichlet problem associated with the degenerate nonlinear elliptic equations

$$\begin{aligned} & -\operatorname{div}[\mathcal{A}(x, \nabla u) \omega_2(x) + \mathcal{B}(x, \nabla u) \nu_1(x)] + \mathcal{H}(x, u, \nabla u) \nu_2 + |u|^{p-2} u \omega_1 \\ & = \rho_0 - \sum_{j=1}^n D_j \rho_j, \\ & u - \psi \in W_0^{1,p}(\Omega, \omega_1, \omega_2), \end{aligned}$$

in the setting of the weighted Sobolev spaces.

1. Introduction. In this paper we prove the existence and uniqueness of (weak) solutions in the weighted Sobolev space $W^{1,p}(\Omega, \omega_1, \omega_2)$ (see Definition 2.2) for the Dirichlet problem

$$(P) \left\{ \begin{array}{l} -\operatorname{div}[\mathcal{A}(x, \nabla u) \omega_2 + \mathcal{B}(x, \nabla u) \nu_1] + \mathcal{H}(x, u, \nabla u) \nu_2 + |u|^{p-2} u \omega_1 \\ = \rho_0 - \sum_{j=1}^n D_j \rho_j, \\ u - \psi \in W_0^{1,p}(\Omega, \omega_1, \omega_2), \end{array} \right.$$

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where $D_j = \partial/\partial x_j$ ($j = 1, \dots, n$), Ω is a bounded open set in $\mathbb{R}^n$, $\omega_1, \omega_2, \nu_1$ and ν_2 , are four weight functions, $\psi \in W^{1,p}(\Omega, \omega_1, \omega_2)$ and the functions, $\mathcal{A} : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $\mathcal{B} : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\mathcal{H} : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ are Caratheodory functions which satisfy the following conditions:

- (H1) $x \mapsto \mathcal{A}(x, \xi)$ is measurable on Ω for all $\xi \in \mathbb{R}^n$,
 $\xi \mapsto \mathcal{A}(x, \xi)$ is continuous on $\mathbb{R}^n$ for almost all $x \in \Omega$.
- (H2) $(\mathcal{A}(x, \xi) - \mathcal{A}(x, \xi')) \cdot (\xi - \xi') > 0$, whenever $\xi, \xi' \in \mathbb{R}^n$, $\xi \neq \xi'$ and
 $\mathcal{A}(x, \xi) = (\mathcal{A}_1(x, \xi), \dots, \mathcal{A}_n(x, \xi))$ (where a dot denotes here the Euclidian scalar product in $\mathbb{R}^n$).
- (H3) $\mathcal{A}(x, \xi) \cdot \xi \geq \lambda_1 |\xi|^p$, where λ_1 is a positive constant and $1 < p < \infty$.
- (H4) $|\mathcal{A}(x, \xi)| \leq h_1(x) |\xi|^{p/p'}$, where h_1 is a nonnegative function and $h_1 \in L^\infty(\Omega)$ (with $1/p + 1/p' = 1$).
- (H5) $x \mapsto \mathcal{B}(x, \xi)$ is measurable on Ω for all $\xi \in \mathbb{R}^n$,
 $\xi \mapsto \mathcal{B}(x, \xi)$ is continuous on $\mathbb{R}^n$ for almost all $x \in \Omega$.
- (H6) $(\mathcal{B}(x, \xi) - \mathcal{B}(x, \xi')) \cdot (\xi - \xi') \geq 0$, whenever $\xi, \xi' \in \mathbb{R}^n$, $\xi \neq \xi'$ and
 $\mathcal{B}(x, \xi) = (\mathcal{B}_1(x, \xi), \dots, \mathcal{B}_n(x, \xi))$.
- (H7) $\mathcal{B}(x, \xi) \cdot \xi \geq \lambda_2 |\xi|^q$, where $\lambda_2 > 0$ is a constant and $1 < q < \infty$.
- (H8) $|\mathcal{B}(x, \xi)| \leq h_2(x) |\xi|^{q/q'}$, $1 < q < \infty$, h_2 is a nonnegative function and
 $h_2 \in L^\infty(\Omega)$, $1/q + 1/q' = 1$.
- (H9) $x \mapsto \mathcal{H}(x, \eta, \xi)$ is measurable on Ω for all $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^n$,
 $(\eta, \xi) \mapsto \mathcal{H}(x, \eta, \xi)$ is continuous on $\mathbb{R} \times \mathbb{R}^n$ for almost all $x \in \Omega$.
- (H10) $[\mathcal{H}(x, \eta, \xi) - \mathcal{H}(x, \eta', \xi')](\eta - \eta') > 0$, whenever $\eta, \eta' \in \mathbb{R}$, $\eta \neq \eta'$.
- (H11) $\mathcal{H}(x, \eta, \xi) \eta \geq \lambda_3 |\xi|^s + \Lambda_3 |\eta|^s$, where λ_3 and Λ_3 are nonnegative constants and $1 < s < \infty$.
- (H12) $|\mathcal{H}(x, \eta, \xi)| \leq h_3(x) |\eta|^{s/s'} + h_4(x) |\xi|^{s/s'}$, where h_3 and h_4 are nonnegative functions, with h_3 and $h_4 \in L^\infty(\Omega)$, $1/s + 1/s' = 1$.

Let Ω be a bounded open set in $\mathbb{R}^n$. By the symbol $\mathcal{W}(\Omega)$ we denote the set of all measurable a.e. in Ω , positive and finite functions $\omega = \omega(x)$, $x \in \Omega$. Elements of $\mathcal{W}(\Omega)$ will be called *weight functions*. Every weight ω gives rise to a measure on the measurable subsets of $\mathbb{R}^n$ through integration. This measure will be denoted by μ_ω . Thus, $\mu_\omega(E) = \int_E \omega(x) dx$ for measurable sets $E \subset \mathbb{R}^n$.

In general, the Sobolev spaces $W^{k,p}(\Omega)$ without weights occur as spaces of solutions for elliptic and parabolic partial differential equations. For degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces (see [3], [4], [5], [8] and [9]). In various applications we can meet boundary value problems for elliptic equations whose ellipticity is disturbed in the sense that some degeneration or singularity appears. There are several very concrete problems from practice which lead to such differential equations, e.g. from glaciology, non-Newtonian fluid mechanics,

flows through porous media, differential geometry, celestial mechanics, climatology, petroleum extraction and reaction-diffusion problems (see some examples of applications of degenerate elliptic equations in [2] and [7]).

A class of weights, which is particularly well understood, is the class of A_p -weights (or Muckenhoupt class) that was introduced by B. Muckenhoupt (see [17]). These classes have found many useful applications in harmonic analysis (see [18]). Another reason for studying A_p -weights is the fact that powers of distance to submanifolds of $\mathbb{R}^n$ often belong to A_p (see [14]). There are, in fact, many interesting examples of weights (see [12] for p -admissible weights).

The following theorem will be proved in Section 3.

Theorem 1.1. *Assume (H1)–(H12) and $\omega_2 \leq \omega_1$.*

(i) *Let $1 < q, s < p < \infty$, ω_1 and ω_2 be A_p -weights, $\nu_1, \nu_2 \in \mathcal{W}(\Omega)$, $\frac{\nu_1}{\omega_2} \in L^{r_1}(\Omega, \omega_2)$ (where $r_1 = p/(p - q)$), $\frac{\nu_2}{\omega_1} \in L^{r_2}(\Omega, \omega_1)$ and $\frac{\nu_2}{\omega_2} \in L^{r_2}(\Omega, \omega_2)$ (where $r_2 = p/(p - s)$).*

(ii) *$\frac{\rho_0}{\omega_1} \in L^{p'}(\Omega, \omega_1)$, $\frac{\rho_j}{\omega_2} \in L^{p'}(\Omega, \omega_2)$ ($j = 1, \dots, n$) and $\psi \in W^{1,p}(\Omega, \omega_1, \omega_2)$.*

Then the problem (P) has a unique solution $u \in W^{1,p}(\Omega, \omega_1, \omega_2)$ with $u - \psi \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$.

The paper is organized as follows. In Section 2 we present the definitions and basics results. In Section 3 we prove our main result about existence and uniqueness of solutions for problem (P).

2. Definitions and basic results. We recall some standards notations, properties and results which will be used throughout the paper.

Let ω be a locally integrable nonnegative function in $\mathbb{R}^n$ and assume that $0 < \omega < \infty$ almost everywhere. We say that ω belongs to the Muckenhoupt class A_p , $1 \leq p < \infty$, or that ω is an A_p -weight, if there is a constant $C = C_{p,\omega}$ (called A_p -constant) such that

$$\begin{aligned} \left(\frac{1}{|B|} \int_B \omega(x) dx \right) \left(\frac{1}{|B|} \int_B \omega^{1/(1-p)}(x) dx \right)^{p-1} &\leq C, \quad \text{when } 1 < p < \infty, \\ \left(\frac{1}{|B|} \int_B \omega(x) dx \right) &\leq C \operatorname{ess\,inf}_B \omega, \quad \text{when } p = 1, \end{aligned}$$

for all balls $B \subset \mathbb{R}^n$, where $|\cdot|$ denotes the n -dimensional Lebesgue measure in $\mathbb{R}^n$. If $1 < q \leq p$, then $A_q \subset A_p$ (see [11], [12] or [18] for more information about A_p -weights). The weight ω satisfies the doubling condition if there exists a positive constant C such that $\mu_\omega(B(x; 2r)) \leq C\mu_\omega(B(x; r))$, for every ball $B = B(x; r) \subset \mathbb{R}^n$, where $\mu_\omega(B) = \int_B \omega(x) dx$. If $\omega \in A_p$, then μ_ω is doubling (see Corollary 15.7 in [12]).

As an example of A_p -weight one can take the function $\omega(x) = |x|^\alpha$, $x \in \mathbb{R}^n$ which is in A_p if and only if $-n < \alpha < n(p - 1)$ (see Corollary 4.4, Chapter IX in [18]). Another example is $\omega(x) = |x|^\alpha (\max\{1, -\ln(|x|)\})^\beta$. This is

an A_1 -weight if and only if $-n < \alpha < 0$ or $\alpha = 0 \leq \beta$ (see Proposition 7.2 in [1]).

If $\omega \in A_p$, $1 < p < \infty$, then

$$\left(\frac{|E|}{|B|}\right)^p \leq C \frac{\mu_\omega(E)}{\mu_\omega(B)},$$

whenever B is a ball in $\mathbb{R}^n$ and E is a measurable subset of B (see 15.5 *strong doubling property* in [12]). Therefore, if $\mu_\omega(E) = 0$ then $|E| = 0$. The measure μ_ω and the Lebesgue measure $|\cdot|$ are mutually absolutely continuous, i.e., they have the same zero sets ($\mu_\omega(E) = 0$ if and only if $|E| = 0$); so there is no need to specify the measure when using the ubiquitous expression almost everywhere and almost every, both abbreviated a.e.

In order to discuss the problem (P) , we need some elementary results for weighted Lebesgue spaces $L^p(\Omega, \omega)$ and the weighted Sobolev spaces $W^{1,p}(\Omega, \omega_1, \omega_2)$ and $W_0^{1,p}(\Omega, \omega_1, \omega_2)$.

Definition 2.1. Let ω be a weight and let $\Omega \subset \mathbb{R}^n$ be a bounded open set. For $1 < p < \infty$ we define $L^p(\Omega, \omega)$ as the set of measurable functions f on Ω such that

$$\|f\|_{L^p(\Omega, \omega)} = \left(\int_\Omega |f|^p \omega \, dx \right)^{1/p} < \infty.$$

We define $L^p(\Omega, \omega; \mathbb{R}^n) = \left\{ \varphi : \Omega \rightarrow \mathbb{R}^n : \int_\Omega |\varphi|^p \omega \, dx < \infty \right\}$. Denote the norm of $L^p(\Omega, \omega)$ and $L^p(\Omega, \omega; \mathbb{R}^n)$ by $\|\cdot\|_{L^p(\Omega, \omega)}$.

If $\omega \in A_p$, $1 < p < \infty$, then $\omega^{-1/(p-1)}$ is locally integrable and $L^p(\Omega, \omega) \subset L_{\text{loc}}^1(\Omega)$ for every open set Ω (see Remark 1.2.4 in [19]). It thus makes sense to talk about weak derivatives of functions in $L^p(\Omega, \omega)$.

Definition 2.2. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let ω_1 and ω_2 be A_p -weights ($1 < p < \infty$). We define the weighted Sobolev space $W^{1,p}(\Omega, \omega_1, \omega_2)$ as the set of functions $u \in L^p(\Omega, \omega_1)$ with weak derivatives $D_j u \in L^p(\Omega, \omega_2)$ (or $\nabla u = (D_1 u, \dots, D_n u) \in L^p(\Omega, \omega_2; \mathbb{R}^n)$, ∇u is the weak gradient of u). The norm of u in $W^{1,p}(\Omega, \omega_1, \omega_2)$ is defined by

$$(2.1) \quad \|u\|_{W^{1,p}(\Omega, \omega_1, \omega_2)} = \left(\int_\Omega |u|^p \omega_1 \, dx + \int_\Omega |\nabla u|^p \omega_2 \, dx \right)^{1/p}.$$

The space $W_0^{1,p}(\Omega, \omega_1, \omega_2)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.1). Equipped with this norm, $W_0^{1,p}(\Omega, \omega_1, \omega_2)$ is a reflexive Banach space (see [8], [15], [16] or [22] for more information about the spaces $W^{1,p}(\Omega, \omega_1, \omega_2)$). The dual of the space $W_0^{1,p}(\Omega, \omega_1, \omega_2)$ is the space

$$\begin{aligned} [W_0^{1,p}(\Omega, \omega_1, \omega_2)]^* &= \left\{ T = f_0 - \operatorname{div}(F), \, F = (f_1, \dots, f_n) : \right. \\ &\quad \left. \frac{f_0}{\omega_1} \in L^{p'}(\Omega, \omega_1), \, \frac{f_j}{\omega_2} \in L^{p'}(\Omega, \omega_2), \, j = 1, \dots, n \right\}. \end{aligned}$$

In this paper we use the following result.

Theorem 2.1. *Let $\omega \in A_p$, $1 < p < \infty$ and let Ω be a bounded open set in $\mathbb{R}^n$. If $u_m \rightarrow u$ in $L^p(\Omega, \omega)$, then there exist a subsequence $\{u_{m_k}\}$ and a function $\Phi \in L^p(\Omega, \omega)$ such that*

- (i) $u_{m_k}(x) \rightarrow u(x)$, $m_k \rightarrow \infty$ a.e. on Ω ;
- (ii) $|u_{m_k}(x)| \leq \Phi(x)$ a.e. on Ω .

Proof. The proof of this theorem follows the lines of the proof of Theorem 2.8.1 in [10]. $\square$

Definition 2.3. We say that an element $u \in W^{1,p}(\Omega, \omega_1, \omega_2)$ is a (weak) solution of equation

$$\begin{aligned} & -\operatorname{div}[\mathcal{A}(x, \nabla u(x))\omega_2(x) + \mathcal{B}(x, \nabla u(x))\nu_1(x)] + \mathcal{H}(x, u, \nabla u)\nu_2 + |u|^{p-2}u\omega_1 \\ & = \rho_0 - \sum_{j=1}^n D_j \rho_j, \end{aligned}$$

if

$$\begin{aligned} & \int_{\Omega} \mathcal{A}(x, \nabla u) \cdot \nabla \varphi \omega_2 dx + \int_{\Omega} \mathcal{B}(x, \nabla u) \cdot \nabla \varphi \nu_1 dx + \int_{\Omega} \mathcal{H}(x, u, \nabla u) \varphi \nu_2 dx \\ & + \int_{\Omega} |u|^{p-2}u \varphi \omega_1 dx = \int_{\Omega} \rho_0 \varphi dx + \sum_{j=1}^n \rho_j D_j \varphi dx \end{aligned}$$

for all $\varphi \in W^{1,p}(\Omega, \omega_1, \omega_2)$, or we can write

$$\begin{aligned} & \int_{\Omega} \left(\mathcal{H}(x, u, \nabla u) \varphi \frac{\nu_2}{\omega_1} + |u|^{p-2}u \varphi - \frac{\rho_0}{\omega_1} \varphi \right) \omega_1 dx \\ & + \int_{\Omega} \left(\mathcal{A}(x, \nabla u) \cdot \nabla \varphi + \mathcal{B}(x, \nabla u) \cdot \nabla \varphi \frac{\nu_1}{\omega_2} - \frac{\rho}{\omega_2} \cdot \nabla \varphi \right) \omega_2 dx = 0, \end{aligned}$$

where $\rho = (\rho_1, \dots, \rho_n)$.

Remark 2.2. (i) If $\frac{\nu_1}{\omega_2} \in L^{r_1}(\Omega, \omega_2)$ (where $r_1 = p/(p-q)$), then $\|u\|_{L^q(\Omega, \nu_1)} \leq C_{1,2} \|u\|_{L^p(\Omega, \omega_2)}$, where $C_{1,2} = \|\nu_1/\omega_2\|_{L^{r_1}(\Omega, \omega_2)}^{1/q}$. In fact, by Hölder's inequality we obtain

$$\begin{aligned} \|u\|_{L^q(\Omega, \nu_1)}^q &= \int_{\Omega} |u|^q \nu_1 dx \\ &= \int_{\Omega} |u|^q \frac{\nu_1}{\omega_2} \omega_2 dx \\ &\leq \left(\int_{\Omega} |u|^{q(p/q)} \omega_2 dx \right)^{q/p} \left(\int_{\Omega} (\nu_1/\omega_2)^{p/(p-q)} \omega_2 dx \right)^{(p-q)/q} \\ &= \|u\|_{L^p(\Omega, \omega_2)}^q \|\nu_1/\omega_2\|_{L^{r_1}(\Omega, \omega_2)}. \end{aligned}$$

(ii) Analogously, if $\frac{\nu_2}{\omega_1} \in L^{r_2}(\Omega, \omega_1)$ and $\frac{\nu_2}{\omega_2} \in L^{r_2}(\Omega, \omega_2)$ (where $r_2 = p/(p-s)$), then

$$\begin{aligned}\|u\|_{L^s(\Omega, \nu_2)} &\leq C_{2,1} \|u\|_{L^p(\Omega, \omega_1)}, \\ \|u\|_{L^s(\Omega, \nu_2)} &\leq C_{2,2} \|u\|_{L^p(\Omega, \omega_2)},\end{aligned}$$

where $C_{2,1} = \|\nu_2/\omega_1\|_{L^{r_2}(\Omega, \omega_1)}^{1/s}$ and $C_{2,2} = \|\nu_2/\omega_2\|_{L^{r_2}(\Omega, \omega_2)}^{1/s}$.

Proposition 2.3. *Let $1 < p < \infty$. There exist two positive constants β_p, γ_p such that for every $x, y \in \mathbb{R}^n$,*

$$\beta_p(|x| + |y|)^{p-2}|x - y|^2 \leq (|x|^{p-2}x - |y|^{p-2}y) \cdot (x - y) \leq \gamma_p(|x| + |y|)^{p-2}|x - y|.$$

Proof. See Proposition 17.3 in [6]. $\square$

In the proof of Theorem 1.1 we will use the following result.

Let X be a reflexive Banach space and denote its dual by X^* . Let $\|\cdot\|_X$ be the norm of X and $\langle \cdot, \cdot \rangle$ be a pairing between X and X^* .

Theorem 2.4. *Let $\mathcal{K}$ be a nonempty closed convex subset of X and let $\mathcal{T} : \mathcal{K} \rightarrow X^*$ be a monotone, coercive and weakly continuous on $\mathcal{K}$. Then there exists an element $u \in \mathcal{K}$ such that $\langle \mathcal{T}u, v - u \rangle \geq 0$ whenever $v \in \mathcal{K}$.*

Proof. See Corollary III.1.8 in [13]. $\square$

For more information on the theory of monotone operators see [21].

3. Main result. Let $X = L^p(\Omega, \omega_1) \times L^p(\Omega, \omega_2; \mathbb{R}^n)$. The norm of X is

$$\|(g, f)\|_X = \|g\|_{L^p(\Omega, \omega_1)} + \|f\|_{L^p(\Omega, \omega_2; \mathbb{R}^n)},$$

for each element $(g, f) \in X$ ($f = (f_1, \dots, f_n)$). Then X is a reflexive Banach space and its dual $X^* = L^{p'}(\Omega, \omega_1) \times L^{p'}(\Omega, \omega_2; \mathbb{R}^n)$. Let $\langle \cdot, \cdot \rangle$ be the usual pairing between X and X^* :

$$\langle (g_1, f), (g_2, h) \rangle = \int_{\Omega} g_1 g_2 \omega_1 dx + \int_{\Omega} f \cdot h \omega_2 dx,$$

(where $f = (f_1, \dots, f_n)$ and $h = (h_1, \dots, h_n)$).

Let $\psi \in W^{1,p}(\Omega, \omega_1, \omega_2)$. Define the set

$$\mathcal{K}_{\psi} = \{(g, \nabla g) : g \in W^{1,p}(\Omega, \omega_1, \omega_2) \text{ and } g - \psi \in W_0^{1,p}(\Omega, \omega_1, \omega_2)\}.$$

Lemma 3.1. $\mathcal{K}_{\psi}$ is a nonempty closed convex subset of X .

Proof. (a) Suppose that $(u, \nabla u) \in \mathcal{K}_{\psi}$. Hence $u \in W^{1,p}(\Omega, \omega_1, \omega_2)$. Then $u \in L^p(\Omega, \omega_1)$ and $\nabla u \in L^p(\Omega, \omega_2; \mathbb{R}^n)$. Therefore, $(u, \nabla u) \in X$. Thus, $\mathcal{K}_{\psi} \subset X$.

(b) If $(u_k, \nabla u_k) \in \mathcal{K}_\psi$ is a sequence which converges to $(g, f) \in X$ ($u_k \rightarrow g$ in $L^p(\Omega, \omega_1)$ and $\nabla u_k \rightarrow f$ in $L^p(\Omega, \omega_2)$, where $f = (f_1, \dots, f_n) \in L^p(\Omega, \omega_2; \mathbb{R}^n)$), then (since $\omega_2 \leq \omega_1$)

$$\begin{aligned} (i) \quad & \int_{\Omega} |u_k - g|^p \omega_2 dx = \int_{\Omega} |(u_k - \psi) - (g - \psi)|^p \omega_2 dx \\ & \leq \int_{\Omega} |(u_k - \psi) - (g - \psi)|^p \omega_1 dx = \int_{\Omega} |u_k - g|^p \omega_1 dx \rightarrow 0, \\ (ii) \quad & \int_{\Omega} |(\nabla u_k - \nabla \psi) - (f - \nabla \psi)|^p \omega_2 dx = \int_{\Omega} |\nabla u_k - f|^p \omega_2 dx \rightarrow 0, \end{aligned}$$

as $k \rightarrow \infty$. Since $\omega_2 \in A_p$, then $\nabla g = f \in L^p(\Omega, \omega_2, \mathbb{R}^n)$ (by the uniqueness of the gradient). And since $u_k - \psi \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$, then $g - \psi \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$. Hence $g \in W^{1,p}(\Omega, \omega_1, \omega_2)$. Therefore, $(g, f) = (g, \nabla g) \in \mathcal{K}_\psi$. Thus, $\mathcal{K}_\psi$ is closed in X .

(c) Let $(u, \nabla u), (v, \nabla v) \in \mathcal{K}_\psi$ and $\alpha \in [0, 1]$. Then $\alpha u + (1 - \alpha)v \in W^{1,p}(\Omega, \omega_1, \omega_2)$ and

$$\alpha u + (1 - \alpha)v - \psi = \alpha(u - \psi) + (1 - \alpha)(v - \psi) \in W_0^{1,p}(\Omega, \omega_1, \omega_2).$$

Hence $\alpha(u, \nabla u) + (1 - \alpha)(v, \nabla v) = (\alpha u + (1 - \alpha)v, \nabla(\alpha u + (1 - \alpha)v)) \in \mathcal{K}_\psi$. Therefore, $\mathcal{K}_\psi$ is convex in X . $\square$

Now, define a mapping $\mathcal{T} : \mathcal{K}_\psi \rightarrow X^*$ by the formula

$$\begin{aligned} (3.1) \quad \mathcal{T}(u, \nabla u) = & \left(\mathcal{H}(x, u, \nabla u) \frac{\nu_2}{\omega_1} + |u|^{p-2}u - \frac{\rho_0}{\omega_1}, \right. \\ & \left. \mathcal{A}(x, \nabla u) + \mathcal{B}(x, \nabla u) \frac{\nu_1}{\omega_2} - \frac{\rho}{\omega_2} \right), \end{aligned}$$

where $\rho = (\rho_1, \dots, \rho_n)$. For convenience, we denote $\mathcal{T}(u, \nabla u)$ simply by $\mathcal{T}(u)$. For each element $(g, f) \in X$, we have

$$\begin{aligned} \langle \mathcal{T}(u), (g, f) \rangle &= \int_{\Omega} \left(\mathcal{H}(x, u, \nabla u) \frac{\nu_2}{\omega_1} + |u|^{p-2}u - \frac{\rho_0}{\omega_1} \right) g \omega_1 dx \\ &+ \int_{\Omega} \left(\mathcal{A}(x, \nabla u) + \mathcal{B}(x, \nabla u) \frac{\nu_1}{\omega_2} - \frac{\rho}{\omega_2} \right) \cdot f \omega_2 dx \\ &= \int_{\Omega} \mathcal{H}(x, u, \nabla u) g \nu_2 dx + \int_{\Omega} |u|^{p-2}u g \omega_1 dx - \int_{\Omega} \rho_0 g dx \\ &+ \int_{\Omega} \mathcal{A}(x, \nabla u) \cdot f \omega_2 dx + \int_{\Omega} \mathcal{B}(x, \nabla u) \cdot f \nu_1 dx - \int_{\Omega} \rho \cdot f dx. \end{aligned}$$

(i) By assumption (H4) we have

$$\begin{aligned}
 (3.2) \quad \left| \int_{\Omega} \mathcal{A}(x, \nabla u) \cdot f \, \omega_2 \, dx \right| &\leq \int_{\Omega} |\mathcal{A}(u, \nabla u)| |f| \, \omega_2 \, dx \\
 &\leq \int_{\Omega} h_1 |\nabla u|^{p-1} |f| \, \omega_2 \, dx \\
 &\leq \|h_1\|_{L^\infty(\Omega)} \|\nabla u\|_{L^p(\Omega, \omega_2)}^{p-1} \|f\|_{L^p(\Omega, \omega_2)} \\
 &\leq \|h_1\|_{L^\infty(\Omega)} \|(u, \nabla u)\|_X^{p-1} \|(g, f)\|_X.
 \end{aligned}$$

(ii) By assumption (H8) and Remark 2.2(i) we obtain

$$\begin{aligned}
 (3.3) \quad \left| \int_{\Omega} \mathcal{B}(x, \nabla u) \cdot f \, \nu_1 \, dx \right| &\leq \int_{\Omega} |\mathcal{B}(x, \nabla u)| |f| \, \nu_1 \, dx \\
 &\leq \int_{\Omega} h_2 |\nabla u|^{q-1} |f| \, \nu_1 \, dx \\
 &\leq \|h_2\|_{L^\infty(\Omega)} \left(\int_{\Omega} |\nabla u|^{(q-1)q'} \, \nu_1 \, dx \right)^{1/q'} \left(\int_{\Omega} |f| \, \nu_1 \, dx \right)^{1/q} \\
 &= \|h_2\|_{L^\infty(\Omega)} \|\nabla u\|_{L^q(\Omega, \nu_1)}^{q-1} \|f\|_{L^q(\Omega, \nu_1)} \\
 &\leq \|h_2\|_{L^\infty(\Omega)} C_{1,2}^{q-1} \|\nabla u\|_{L^p(\Omega, \omega_2)}^{q-1} C_{1,2} \|f\|_{L^p(\Omega, \omega_2)} \\
 &\leq C_{1,2}^q \|h_2\|_{L^\infty(\Omega)} \|(u, \nabla u)\|_X^{q-1} \|(g, f)\|_X.
 \end{aligned}$$

(iii) By (H12) and Remark 2.2(ii) we get

$$\begin{aligned}
 (3.4) \quad \left| \int_{\Omega} \mathcal{H}(x, u, \nabla u) g \, \nu_2 \, dx \right| &\leq \int_{\Omega} |\mathcal{H}(x, u, \nabla u)| |g| \, \nu_2 \, dx \\
 &\leq \int_{\Omega} \left(h_3 |u|^{s-1} + h_4 |\nabla u|^{s-1} \right) |g| \, \nu_2 \, dx \\
 &\leq \|h_3\|_{L^\infty(\Omega)} \|u\|_{L^s(\Omega, \nu_2)}^{s-1} \|g\|_{L^s(\Omega, \nu_2)} \\
 &\quad + \|h_4\|_{L^\infty(\Omega)} \|\nabla u\|_{L^s(\Omega, \nu_2)}^{s-1} \|g\|_{L^s(\Omega, \nu_2)} \\
 &\leq \|h_3\|_{L^\infty(\Omega)} C_{2,1}^{s-1} \|u\|_{L^p(\Omega, \omega_1)}^{s-1} C_{2,1} \|g\|_{L^p(\Omega, \omega_1)} \\
 &\quad + \|h_4\|_{L^\infty(\Omega)} C_{2,2}^{s-1} \|\nabla u\|_{L^p(\Omega, \omega_2)}^{s-1} C_{2,1} \|g\|_{L^p(\Omega, \omega_1)} \\
 &\leq C \|(u, \nabla u)\|_X^{s-1} \|(g, f)\|_X,
 \end{aligned}$$

where $C = \max\{C_{2,1}^s \|h_3\|_{L^\infty(\Omega)}, C_{2,1} C_{2,2}^{s-1} \|h_4\|_{L^\infty(\Omega)}\}$.

(iv) We also have

$$(3.5) \quad \left| \int_{\Omega} |u|^{p-2} u g \omega_1 dx \right| \leq \int_{\Omega} |u|^{p-1} |g| \omega_1 dx \leq \|u\|_{L^p(\Omega, \omega_1)}^{p-1} \|g\|_{L^p(\Omega, \omega_1)} \\ \leq \|(u, \nabla u)\|_X^{p-1} \|(g, f)\|_X.$$

(v) Moreover,

$$(3.6) \quad \left| \int_{\Omega} \rho_0 g dx + \int_{\Omega} \rho \cdot f dx \right| \\ \leq \int_{\Omega} \frac{|\rho_0|}{\omega_1} g \omega_1 dx + \int_{\Omega} \frac{|\rho|}{\omega_2} |f| \omega_2 dx \\ \leq \|\rho_0/\omega_1\|_{L^{p'}(\Omega, \omega_1)} \|g\|_{L^p(\Omega, \omega_1)} + \|\rho/\omega_2\|_{L^{p'}(\Omega, \omega_2)} \|f\|_{L^p(\Omega, \omega_2)} \\ \leq (\|\rho_0/\omega_1\|_{L^{p'}(\Omega, \omega_1)} + \|\rho/\omega_2\|_{L^{p'}(\Omega, \omega_2)}) \|(g, f)\|_X.$$

Therefore, by (i), (ii), (iii), (iv) and (v), $\mathcal{T}(u) \in X^*$ for each $(u, \nabla u) \in \mathcal{K}_{\psi}$ and the mapping $\mathcal{T}$ is well defined.

Lemma 3.2. *The mapping $\mathcal{T}$ defined in (3.1) is monotone and coercive.*

Proof. (I) If $(u, \nabla u), (f, \nabla f) \in \mathcal{K}_{\psi}$, then

$$(3.7) \quad \mathcal{T}(u) - \mathcal{T}(f) \\ = \left(\mathcal{H}(x, u, \nabla u) \frac{\nu_2}{\omega_1} + |u|^{p-2} u - \frac{\rho_0}{\omega_1}, \mathcal{A}(x, \nabla u) + \mathcal{B}(x, \nabla u) \frac{\nu_1}{\omega_2} - \frac{\rho}{\omega_2} \right) \\ - \left(\mathcal{H}(x, f, \nabla f) \frac{\nu_2}{\omega_1} + |f|^{p-2} f - \frac{\rho_0}{\omega_1}, \mathcal{A}(x, \nabla f) + \mathcal{B}(x, \nabla f) \frac{\nu_1}{\omega_2} - \frac{\rho}{\omega_2} \right) \\ = \left(\mathcal{H}(x, u, \nabla u) \frac{\nu_2}{\omega_1} - \mathcal{H}(x, f, \nabla f) \frac{\nu_2}{\omega_1} + |u|^{p-2} u - |f|^{p-2} f, \right. \\ \left. \mathcal{A}(x, \nabla u) - \mathcal{A}(x, \nabla f) + \mathcal{B}(x, \nabla u) \frac{\nu_1}{\omega_2} - \mathcal{B}(x, \nabla f) \frac{\nu_1}{\omega_2} \right).$$

Then by assumptions (H2), (H6) and (H10) we have

$$\langle \mathcal{T}(u) - \mathcal{T}(f), (u, \nabla u) - (f, \nabla f) \rangle \\ = \int_{\Omega} (\mathcal{H}(x, u, \nabla u) - \mathcal{H}(x, f, \nabla f))(u - f) \nu_2 dx \\ + \int_{\Omega} (|u|^{p-2} u - |f|^{p-2} f)(u - f) \omega_1 dx \\ + \int_{\Omega} (\mathcal{A}(x, \nabla u) - \mathcal{A}(x, \nabla f)) \cdot \nabla(u - f) \omega_2 dx \\ + \int_{\Omega} \left(\mathcal{B}(x, \nabla u) - \mathcal{B}(x, \nabla f) \right) \cdot (\nabla u - \nabla f) \nu_1 dx > 0,$$

since, by Proposition 2.3,

$$(|u|^{p-2}u - |f|^{p-2}f)(u - f) \geq \beta_p(|u| + |f|)^{p-2}|u - f| > 0$$

(if $u \neq f$). Hence, $\mathcal{T}$ is monotone.

(II) Let $(f, \nabla f) \in \mathcal{K}_\psi$ be fixed. For each $(u, \nabla u) \in \mathcal{K}_\psi$, by assumptions (H6) and (H10), we have

$$\begin{aligned}
 & \langle T(u) - T(f), (u, \nabla u) - (f, \nabla f) \rangle \\
 &= \int_{\Omega} (\mathcal{H}(x, u, \nabla u) - \mathcal{H}(x, f, \nabla f))(u - f) \nu_2 dx \\
 & \quad + \int_{\Omega} (|u|^{p-2}u - |f|^{p-2}f)(u - f) \omega_1 dx \\
 & \quad + \int_{\Omega} (\mathcal{A}(x, \nabla u) - \mathcal{A}(x, \nabla f)) \cdot \nabla(u - f) \omega_2 dx \\
 (3.8) \quad & \quad + \int_{\Omega} \left(\mathcal{B}(x, \nabla u) - \mathcal{B}(x, \nabla f) \right) \cdot (\nabla u - \nabla f) \nu_1 dx \\
 & > \int_{\Omega} (|u|^{p-2}u - |f|^{p-2}f)(u - f) \omega_1 dx \\
 & \quad + \int_{\Omega} (\mathcal{A}(x, \nabla u) - \mathcal{A}(x, \nabla f)) \cdot \nabla(u - f) \omega_2 dx.
 \end{aligned}$$

By (H3) and (3.2), we obtain

$$\begin{aligned}
 & \int_{\Omega} (\mathcal{A}(x, \nabla u) - \mathcal{A}(x, \nabla f)) \cdot \nabla(u - f) \omega_2 dx \\
 &= \int_{\Omega} \mathcal{A}(x, \nabla u) \cdot \nabla u \omega_2 dx + \int_{\Omega} \mathcal{A}(x, \nabla f) \cdot \nabla f \omega_2 dx \\
 & \quad - \int_{\Omega} \mathcal{A}(x, \nabla u) \cdot \nabla f \omega_2 dx - \int_{\Omega} \mathcal{A}(x, \nabla f) \cdot \nabla u \omega_2 dx \\
 (3.9) \quad & \geq \lambda_1 \int_{\Omega} |\nabla u|^p \omega_2 dx + \lambda_1 \int_{\Omega} |\nabla f|^p \omega_2 dx \\
 & \quad - \int_{\Omega} \mathcal{A}(x, \nabla u) \cdot \nabla f \omega_2 dx - \int_{\Omega} \mathcal{A}(x, \nabla f) \cdot \nabla u \omega_2 dx \\
 & \geq \lambda_1 \int_{\Omega} |\nabla u|^p \omega_2 dx + \lambda_1 \int_{\Omega} |\nabla f|^p \omega_2 dx \\
 & \quad - \|h_1\|_{L^\infty(\Omega)} \|\nabla u\|_{L^p(\Omega, \omega_2)}^{p-1} \|\nabla f\|_{L^p(\Omega, \omega_2)} \\
 & \quad - \|h_1\|_{L^\infty(\Omega)} \|\nabla f\|_{L^p(\Omega, \omega_2)}^{p-1} \|\nabla u\|_{L^p(\Omega, \omega_2)}.
 \end{aligned}$$

Moreover, by (3.5) we have

$$\begin{aligned}
 & \int_{\Omega} (|u|^{p-2}u - |f|^{p-2}f)(u - f) \omega_1 dx \\
 &= \int_{\Omega} |u|^p \omega_1 dx + \int_{\Omega} |f|^p \omega_1 dx \\
 (3.10) \quad & - \int_{\Omega} |u|^{p-2}u f \omega_1 dx - \int_{\Omega} |f|^{p-2}f u \omega_1 dx \\
 & \geq \int_{\Omega} |u|^p \omega_1 dx + \int_{\Omega} |f|^p \omega_1 dx \\
 & - \|u\|_{L^p(\Omega, \omega_1)}^{p-1} \|f\|_{L^p(\Omega, \omega_1)} - \|f\|_{L^p(\Omega, \omega_1)}^{p-1} \|u\|_{L^p(\Omega, \omega_1)}.
 \end{aligned}$$

Hence, by (3.8), (3.9) and (3.10) we obtain

$$\begin{aligned}
 & \langle \mathcal{T}(u) - \mathcal{T}(f), (u, \nabla u) - (f, \nabla f) \rangle \\
 & \geq C_1 \left(\|u\|_{L^p(\Omega, \omega_1)}^p + \|\nabla u\|_{L^p(\Omega, \omega_2)}^p + \|f\|_{L^p(\Omega, \omega_1)}^p + \|\nabla f\|_{L^p(\Omega, \omega_2)}^p \right) \\
 (3.11) \quad & - C_2 \left(\|\nabla u\|_{L^p(\Omega, \omega_2)}^{p-1} \|\nabla f\|_{L^p(\Omega, \omega_2)} + \|\nabla f\|_{L^p(\Omega, \omega_2)}^{p-1} \|\nabla u\|_{L^p(\Omega, \omega_2)} \right. \\
 & \left. + \|u\|_{L^p(\Omega, \omega_1)}^{p-1} \|f\|_{L^p(\Omega, \omega_1)} + \|f\|_{L^p(\Omega, \omega_1)}^{p-1} \|u\|_{L^p(\Omega, \omega_1)} \right),
 \end{aligned}$$

where $C_1 = \min\{1, \lambda_1\}$ and $C_2 = \max\{1, \|h_1\|_{L^\infty(\Omega)}\}$. To estimate the right-hand side of (3.11) from below, we use the inequality $(\sum_{j=1}^4 c_j)^q \leq 4^q \sum_{j=1}^4 c_j^q$ for all $c_j \geq 0$ ($j = 1, 2, 3, 4$) and $q > 0$. We have

$$\begin{aligned}
 (a) \quad & (\|(u, \nabla u)\|_X + \|(f, \nabla f)\|_X)^p \\
 &= (\|u\|_{L^p(\Omega, \omega_1)} + \|\nabla u\|_{L^p(\Omega, \omega_2)} + \|f\|_{L^p(\Omega, \omega_1)} + \|\nabla f\|_{L^p(\Omega, \omega_2)})^p \\
 &\leq 4^p \left(\|u\|_{L^p(\Omega, \omega_1)}^p + \|\nabla u\|_{L^p(\Omega, \omega_2)}^p + \|f\|_{L^p(\Omega, \omega_1)}^p + \|\nabla f\|_{L^p(\Omega, \omega_2)}^p \right); \\
 (b) \quad & \|\nabla u\|_{L^p(\Omega, \omega_2)}^{p-1} \|\nabla f\|_{L^p(\Omega, \omega_2)} + \|u\|_{L^p(\Omega, \omega_1)}^{p-1} \|f\|_{L^p(\Omega, \omega_1)} \\
 &\leq \|(u, \nabla u)\|_X^{p-1} \|(f, \nabla f)\|_X \\
 &\leq (\|(u, \nabla u)\|_X + \|(f, \nabla f)\|_X)^{p-1} \|(f, \nabla f)\|_X; \\
 (c) \quad & \|\nabla u\|_{L^p(\Omega, \omega_2)} \|\nabla f\|_{L^p(\Omega, \omega_2)}^{p-1} + \|u\|_{L^p(\Omega, \omega_1)} \|f\|_{L^p(\Omega, \omega_1)}^{p-1} \\
 &\leq \|(u, \nabla u)\|_X \|(f, \nabla f)\|_X^{p-1} \\
 &\leq (\|(u, \nabla u)\|_X + \|(f, \nabla f)\|_X) \|(f, \nabla f)\|_X^{p-1}.
 \end{aligned}$$

Now using (a), (b) and (c), from (3.11) we obtain

$$\begin{aligned} & \langle \mathcal{T}(u) - \mathcal{T}(f), (u, \nabla u) - (f, \nabla f) \rangle \\ & \geq \frac{1}{4^p} C_1 (\|(u, \nabla u)\|_X + \|(f, \nabla f)\|_X)^p \\ & \quad - C_2 (\|(u, \nabla u)\|_X + \|(f, \nabla f)\|_X)^{p-1} \|(f, \nabla f)\|_X \\ & \quad - C_2 (\|(u, \nabla u)\|_X + \|(f, \nabla f)\|_X) \|(f, \nabla f)\|_X^{p-1}. \end{aligned}$$

Thus,

$$\begin{aligned} & \frac{\langle \mathcal{T}(u) - \mathcal{T}(f), (u, \nabla u) - (f, \nabla f) \rangle}{\|(u, \nabla u) - (f, \nabla f)\|_X} \\ & \geq \frac{\langle \mathcal{T}(u) - \mathcal{T}(f), (u, \nabla u) - (f, \nabla f) \rangle}{\|(u, \nabla u)\|_X + \|(f, \nabla f)\|_X} \\ (3.12) \quad & \geq \frac{1}{4^p} C_1 (\|(u, \nabla u)\|_X + \|(f, \nabla f)\|_X)^{p-1} \\ & \quad - C_2 (\|(u, \nabla u)\|_X + \|(f, \nabla f)\|_X)^{p-2} \|(f, \nabla f)\|_X \\ & \quad - C_2 \|(f, \nabla f)\|_X^{p-1}. \end{aligned}$$

For each sequence $(u_k, \nabla u_k) \in \mathcal{K}_\psi$ with $\|(u_k, \nabla u_k)\|_X \rightarrow \infty$, we have

$$\|(u_k, \nabla u_k)\|_X + \|(f, \nabla f)\|_X \geq \|(u_k, \nabla u_k)\|_X \rightarrow \infty.$$

It follows that

$$\begin{aligned} & \frac{1}{4^p} C_1 (\|(u_k, \nabla u_k)\|_X + \|(f, \nabla f)\|_X)^{p-1} \\ & \quad - C_2 (\|(u_k, \nabla u_k)\|_X + \|(f, \nabla f)\|_X)^{p-2} \|(f, \nabla f)\|_X \\ (3.13) \quad & = \left(\frac{1}{4^p} C_1 - C_2 \frac{\|(f, \nabla f)\|_X}{\|(u_k, \nabla u_k)\|_X + \|(f, \nabla f)\|_X} \right) \\ & \quad \times (\|(u_k, \nabla u_k)\|_X + \|(f, \nabla f)\|_X)^{p-1} \rightarrow \infty, \text{ as } k \rightarrow 0. \end{aligned}$$

Combining (3.12) and (3.13), we obtain

$$\frac{\langle \mathcal{T}(u_k) - \mathcal{T}(f), (u_k, \nabla u_k) - (f, \nabla f) \rangle}{\|(u_k, \nabla u_k) - (f, \nabla f)\|_X} \rightarrow \infty, \text{ as } k \rightarrow \infty.$$

Therefore, $\mathcal{T}$ is coercive in $\mathcal{K}_\psi$. □

Lemma 3.3. *The mapping $\mathcal{T}$ defined in (3.1) is weakly continuous.*

Proof. If $(u, \nabla u), (f, \nabla f) \in \mathcal{K}_\psi$, then

$$\begin{aligned} & \langle \mathcal{T}(u) - \mathcal{T}(f), (f, \nabla f) \rangle \\ & = \int_{\Omega} (\mathcal{B}(x, \nabla u) - \mathcal{B}(x, \nabla f)) \cdot \nabla f \nu_1 dx + \int_{\Omega} (\mathcal{A}(x, \nabla u) - \mathcal{A}(x, \nabla f)) \cdot \nabla f \omega_2 dx \\ & \quad + \int_{\Omega} (\mathcal{H}(x, u, \nabla u) - \mathcal{H}(x, f, \nabla f)) f \nu_2 dx + \int_{\Omega} (|u|^{p-2} u - |f|^{p-2} f) f \omega_1 dx. \end{aligned}$$

The mapping $\mathcal{T}$ is weakly continuous on $\mathcal{K}_\psi$ if $\mathcal{T}(u_m)$ converges to $\mathcal{T}(u)$ weakly in X^* , i.e., $\langle \mathcal{T}(u_m), (f, \nabla f) \rangle \rightarrow \langle \mathcal{T}(u), (f, \nabla f) \rangle$ whenever $(u_m, \nabla u_m), (u, \nabla u) \in \mathcal{K}_\psi$, $(u_m, \nabla u_m) \rightarrow (u, \nabla u)$ in X . It suffices to prove that $\langle \mathcal{T}(u_m) - \mathcal{T}(u), (f, \nabla f) \rangle \rightarrow 0$, for $(f, \nabla f) \in X$.

(a) We define the operators $G_j : \mathcal{K}_\psi \rightarrow L^{q'}(\Omega, \nu_1)$ (for $j = 1, 2, \dots, n$) by the formula

$$(G_j u)(x) = \mathcal{B}_j(x, \nabla u).$$

We now show that the operator G_j is bounded and continuous.

(i) By (H8) and Remark 2.2(i) we have

$$\begin{aligned} \|G_j u\|_{L^{q'}(\Omega, \nu_1)}^{q'} &= \int_{\Omega} |G_j u(x)|^{q'} \nu_1 dx \\ &= \int_{\Omega} |\mathcal{B}_j(x, \nabla u)|^{q'} \nu_1 dx \\ &\leq \int_{\Omega} \left(h_2 |\nabla u|^{q/q'} \right)^{q'} \nu_1 dx \\ &\leq \|h_2\|_{L^\infty(\Omega)}^{q'} \int_{\Omega} |\nabla u|^q \nu_1 dx \\ &\leq C_{1,2}^q \|h_2\|_{L^\infty(\Omega)}^{q'} \left(\int_{\Omega} |\nabla u|^p \omega_2 dx \right)^{q/p} \\ &= C_{1,2}^q \|h_2\|_{L^\infty(\Omega)}^{q'} \|\nabla u\|_{L^p(\Omega, \omega_2)}^q \\ &\leq C_{1,2}^q \|h_2\|_{L^\infty(\Omega)}^{q'} \|(u, \nabla u)\|_X^q. \end{aligned}$$

Hence, G_j is bounded.

(ii) If $(u_m, \nabla u_m) \rightarrow (u, \nabla u) \in \mathcal{K}_\psi$, then $\nabla u_m \rightarrow \nabla u$ in $L^p(\Omega, \omega_2)$. By Theorem 2.1 there exists a subsequence $\{u_{m_k}\}$ and functions $\Phi_1 \in L^p(\Omega, \omega_1)$, $\Phi_2 \in L^p(\Omega, \omega_2)$ such that

$$\begin{aligned} (3.14) \quad &u_{m_k}(x) \rightarrow u(x) \text{ a.e. in } \Omega; \\ &|u_{m_k}(x)| \leq \Phi_1(x) \text{ a.e. in } \Omega; \\ &D_j u_{m_k} \rightarrow D_j u(x) \text{ a.e. in } \Omega; \\ &|\nabla u_{m_k}(x)| \leq \Phi_2(x) \text{ a.e. in } \Omega. \end{aligned}$$

Next, applying (H8), we obtain

$$\begin{aligned} |G_j u_{m_k} - G_j u|^{q'} \nu_1 &= |\mathcal{B}_j(x, \nabla u_{m_k}) - \mathcal{B}_j(x, \nabla u)|^{q'} \nu_1 \\ &\leq C_q (|\mathcal{B}_j(x, \nabla u_{m_k})|^{q'} + |\mathcal{B}_j(x, \nabla u)|^{q'}) \nu_1 dx \\ &\leq C_q [(h_2 |\nabla u_{m_k}|^{q-1})^{q'} + (h_2 |\nabla u|^{q-1})^{q'}] \nu_1 \\ &\leq 2C_q \|h_2\|_{L^\infty(\Omega)}^{q'} \Phi_2^q \nu_1 \in L^1(\Omega), \end{aligned}$$

where C_q depends only on q and by Remark 2.2(i),

$$\int_{\Omega} \Phi_2^q \nu_1 dx \leq C_{1,2}^q \left(\int_{\Omega} \Phi_2^p \omega_2 dx \right)^{q/p}.$$

By condition (H5), we have $G_j u_{m_k}(x) = \mathcal{B}_j(x, \nabla u_{m_k}) \rightarrow \mathcal{B}_j(x, \nabla u) = G_j u(x)$, as $m_k \rightarrow \infty$. Therefore, by the Lebesgue Dominated Convergence Theorem, we obtain

$$\|G_j u_{m_k} - G_j u\|_{L^{q'}(\Omega, \nu_1)} \rightarrow 0,$$

that is, $G u_{m_k} \rightarrow G u$ in $L^{q'}(\Omega, \nu_1)$. From the Convergence Principle in Banach spaces (see Proposition 10.13 in [20]) we conclude that

$$(3.15) \quad G_j u_m \rightarrow G_j u \quad \text{in } L^{q'}(\Omega, \nu_1).$$

(b) We define the operators $F_j : \mathcal{K}_{\psi} \rightarrow L^{p'}(\Omega, \omega_2)$ ($j = 1, 2, \dots, n$) by the formula

$$(F_j u)(x) = \mathcal{A}_j(x, \nabla u).$$

We show that this operator is bounded and continuous.

(i) Using (H4), we obtain

$$\begin{aligned} \|F_j u\|_{L^{p'}(\Omega, \omega_2)}^{p'} &= \int_{\Omega} |F_j u(x)|^{p'} \omega_2 dx \\ &= \int_{\Omega} |\mathcal{A}_j(x, \nabla u)|^{p'} \omega_2 dx \\ &\leq \int_{\Omega} \left(h_1 |\nabla u|^{p/p'} \right)^{p'} \omega_2 dx \\ &\leq \|h_1\|_{L^{\infty}(\Omega)}^{p'} \int_{\Omega} |\nabla u|^p \omega_2 dx \\ &= \|h_1\|_{L^{\infty}(\Omega)}^{p'} \|\nabla u\|_{L^p(\Omega, \omega_2)}^p \\ &\leq \|h_1\|_{L^{\infty}(\Omega)}^{p'} \|(u, \nabla u)\|_X^p. \end{aligned}$$

Hence, F_j is bounded.

(ii) If $(u_m, \nabla u_m) \rightarrow (u, \nabla u) \in \mathcal{K}_{\psi}$, then $\nabla u_m \rightarrow \nabla u$ in $L^p(\Omega, \omega_2)$. By (3.14) and (H4) we obtain

$$\begin{aligned} |F_j u_{m_k} - F_j u|^{p'} \omega_2 &= |\mathcal{A}_j(x, \nabla u_{m_k}) - \mathcal{A}_j(x, \nabla u)|^{p'} \omega_2 \\ &\leq C_p \left(|\mathcal{A}_j(x, \nabla u_{m_k})|^{p'} + |\mathcal{A}_j(x, \nabla u)|^{p'} \right) \omega_2 \\ &\leq C_p \left(h_1^{p'} |\nabla u_{m_k}|^p + h_1^{p'} |\nabla u|^p \right) \omega_2 \\ &\leq 2C_p \|h_1\|_{L^{\infty}(\Omega)}^{p'} \Phi_2^p \omega_2 \in L^1(\Omega), \end{aligned}$$

where C_p depends only on p .

By condition (H1) we have $F_j u_{m_k}(x) = \mathcal{A}_j(x, \nabla u_{m_k}) \rightarrow \mathcal{A}_j(x, \nabla u) = F_j u(x)$, as $m_k \rightarrow \infty$. Therefore, by the Lebesgue Dominated Convergence Theorem, we obtain

$$\|F_j u_{m_k} - F_j u\|_{L^{p'}(\Omega, \omega_2)} \rightarrow 0,$$

that is, $F_j u_{m_k} \rightarrow F_j u$ in $L^{p'}(\Omega, \omega_2)$. From the Convergence Principle in Banach spaces (see Proposition 10.13 in [20]) we conclude that

$$(3.16) \quad F_j u_m \rightarrow F_j u \quad \text{in } L^{p'}(\Omega, \omega_2).$$

(c) We define the operator $H : \mathcal{K}_\psi \rightarrow L^{s'}(\Omega, \nu_2)$ by the formula

$$(Hu)(x) = \mathcal{H}(x, u, \nabla u).$$

(i) Using (H12) and Remark 2.2(i), we have

$$\begin{aligned} \|Hu\|_{L^{s'}(\Omega, \nu_2)}^{s'} &= \int_{\Omega} |Hu|^{s'} \nu_2 dx = \int_{\Omega} |\mathcal{H}(x, u, \nabla u)|^{s'} \nu_2 dx \\ &\leq \int_{\Omega} \left(h_3 |u|^{s/s'} + h_4 |\nabla u|^{s/s'} \right)^{s'} \nu_2 dx \\ &\leq C_s \int_{\Omega} \left(h_3^{s'} |u|^s + h_4^{s'} |\nabla u|^s \right) \nu_2 dx \\ &\leq C_s \left(\|h_3\|_{L^\infty(\Omega)}^{s'} \int_{\Omega} |u|^s \nu_2 dx + \|h_4\|_{L^\infty(\Omega)} \int_{\Omega} |\nabla u|^s \nu_2 dx \right) \\ &\leq C_s \left[\|h_3\|_{L^\infty(\Omega)}^{s'} C_{2,1}^s \left(\int_{\Omega} |u|^p \omega_1 dx \right)^{s/p} \right. \\ &\quad \left. + \|h_4\|_{L^\infty(\Omega)}^{s'} C_{2,2}^s \left(\int_{\Omega} |\nabla u|^p \omega_2 dx \right)^{s/p} \right] \\ &\leq C_s C \left[\left(\int_{\Omega} |u|^p \omega_1 dx \right)^{s/p} + \left(\int_{\Omega} |\nabla u|^p \omega_2 dx \right)^{s/p} \right] \\ &\leq C_s C \|(u, \nabla u)\|_X^s, \end{aligned}$$

where $C = \max\{C_{2,1}^s \|h_3\|_{L^\infty(\Omega)}^{s'}, C_{2,2}^s \|h_4\|_{L^\infty(\Omega)}^{s'}\}$.

(ii) If $(u_m, \nabla u_m) \rightarrow (u, \nabla u) \in \mathcal{K}_\psi$, analogously to what was demonstrated with the operators G , F_j and by (3.14),

$$\begin{aligned} |H(u_{m_k}) - H(u)|^{s'} \nu_2 &= |\mathcal{H}(x, u_{m_k}, \nabla u_{m_k}) - \mathcal{H}(x, u, \nabla u)|^{s'} \nu_2 \\ &\leq C_s \left(|\mathcal{H}(x, u_{m_k}, \nabla u_{m_k})|^{s'} + |\mathcal{H}(x, u, \nabla u)|^{s'} \right) \nu_2 \end{aligned}$$

$$\begin{aligned}
&\leq C_s \left[\left(h_3 |u_{m_k}|^{s/s'} + h_4 |\nabla u_{m_k}|^{s/s'} \right)^{s'} + \left(h_3 |u|^{s/s'} + h_4 |\nabla u|^{s/s'} \right)^{s'} \right] \nu_2 \\
&\leq C_s \left[\left(\|h_3\|_{L^\infty(\Omega)}^{s'} |u_{m_k}|^s + \|h_4\|_{L^\infty(\Omega)}^{s'} |\nabla u_{m_k}|^s \right) \right. \\
&\quad \left. + \left(\|h_3\|_{L^\infty(\Omega)}^{s'} |u|^s + \|h_4\|_{L^\infty(\Omega)}^{s'} |\nabla u|^s \right) \right] \nu_2 \\
&\leq 2C_s (\|h_3\|_{L^\infty(\Omega)}^{s'} \Phi_1^s + \|h_4\|_{L^\infty(\Omega)}^{s'} \Phi_2^s) \nu_2 \in L^1(\Omega),
\end{aligned}$$

since, by Remark 2.2(ii),

$$\int_{\Omega} \Phi_1^s \nu_2 dx \leq C_{2,1}^s \left(\int_{\Omega} \Phi_1^p \omega_1 dx \right)^{s/p} \quad \text{and} \quad \int_{\Omega} \Phi_2^s \nu_2 dx \leq C_{2,2}^s \left(\int_{\Omega} \Phi_2^p \omega dx \right)^{s/p}.$$

By condition (H9) we have $Hu_{m_k}(x) = \mathcal{H}(x, u_{m_k}, \nabla u_{m_k}) \rightarrow \mathcal{H}(x, u, \nabla u) = Hu(x)$, as $m_k \rightarrow \infty$. Therefore, by the Lebesgue Dominated Convergence Theorem, we obtain

$$\|Hu_{m_k} - Hu\|_{L^{s'}(\Omega, \nu_2)} \rightarrow 0,$$

that is, $Hu_{m_k} \rightarrow Hu$ in $L^{s'}(\Omega, \nu_2)$. From the Convergence Principle in Banach spaces (see Proposition 10.13 in [20]) we conclude that

$$(3.17) \quad Hu_m \rightarrow Hu \quad \text{in} \quad L^{s'}(\Omega, \nu_2).$$

(d) We define the operator $J: \mathcal{K}_\psi \rightarrow L^{p'}(\Omega, \omega_1)$ by the formula

$$(Ju)(x) = |u(x)|^{p-2} u(x).$$

(i) If $(u, \nabla u) \in \mathcal{K}_\psi$, then

$$\begin{aligned}
\|Ju\|_{L^{p'}(\Omega, \omega_1)}^{p'} &= \int_{\Omega} |Ju(x)|^{p'} \omega_1 dx \\
&= \int_{\Omega} |u|^{(p-1)p'} \omega_1 dx \\
&= \int_{\Omega} |u|^p \omega_1 dx \\
&\leq \|(u, \nabla u)\|_X^p.
\end{aligned}$$

(ii) If $(u_m, \nabla u_m) \rightarrow (u, \nabla u) \in \mathcal{K}_\psi$, analogously to what was demonstrated with the operators G , F_j , H and by (3.14), we obtain

$$(3.18) \quad Ju_m \rightarrow Ju \quad \text{in} \quad L^{p'}(\Omega, \omega_1).$$

Therefore, if $(u_m, \nabla u_m) \rightarrow (u, \nabla u)$ in $\mathcal{K}_\psi$, by Remark 2.2(i), (ii) we have

$$\begin{aligned}
 & |\langle \mathcal{T}(u_m) - \mathcal{T}(u), (f, \nabla f) \rangle| \\
 &= \left| \sum_{j=1}^n \int_{\Omega} \left(\mathcal{B}_j(x, \nabla u_m) - \mathcal{B}_j(x, \nabla u) \right) D_j f \nu_1 dx \right. \\
 &\quad + \sum_{j=1}^n \int_{\Omega} \left(\mathcal{A}_j(x, \nabla u_m) - \mathcal{A}_j(x, \nabla u) \right) D_j f \omega_2 dx \\
 &\quad + \int_{\Omega} \left(\mathcal{H}(x, u_m, \nabla u_m) - \mathcal{H}(x, u, \nabla u) \right) f \nu_2 dx \\
 &\quad \left. + \int_{\Omega} (|u_m|^{p-2} u_m - |u|^{p-2} u) f \omega_1 dx \right| \\
 &\leq \sum_{j=1}^n \int_{\Omega} |G_j u_m - G_j u| |D_j f| \nu_1 dx + \sum_{j=1}^n \int_{\Omega} |F_j u_m - F_j u| |D_j f| \omega_2 dx \\
 &\quad + \int_{\Omega} |H u_m - H u| |f| \nu_2 dx + \int_{\Omega} |J u_m - J u| |f| \omega_1 dx \\
 &\leq \sum_{j=1}^n \|G_j u_m - G_j u\|_{L^{q'}(\Omega, \nu_1)} \|D_j f\|_{L^q(\Omega, \nu_1)} \\
 &\quad + \sum_{j=1}^n \|F_j u_m - F_j u\|_{L^{p'}(\Omega, \omega_2)} \|D_j f\|_{L^p(\Omega, \omega_2)} \\
 &\quad + \|H u_m - H u\|_{L^{s'}(\Omega, \nu_2)} \|f\|_{L^s(\Omega, \nu_2)} + \|J u_m - J u\|_{L^{p'}(\Omega, \omega_1)} \|f\|_{L^p(\Omega, \omega_1)} \\
 &\leq C_{1,2} \left(\sum_{j=1}^n \|G_j u_m - G_j u\|_{L^{q'}(\Omega, \nu_1)} \right) \|\nabla f\|_{L^p(\Omega, \omega_2)} \\
 &\quad + \left(\sum_{j=1}^n \|F_j u_m - F_j u\|_{L^{p'}(\Omega, \omega_2)} \right) \|\nabla f\|_{L^p(\Omega, \omega_2)} \\
 &\quad + C_{2,1} \|H u_m - H u\|_{L^{s'}(\Omega, \nu_2)} \|f\|_{L^p(\Omega, \omega_1)} + \|J u_m - J u\|_{L^{p'}(\Omega, \omega_1)} \|f\|_{L^p(\Omega, \omega_1)} \\
 &\leq \left(C_{1,2} \sum_{j=1}^n \|G_j u_m - G_j u\|_{L^{q'}(\Omega, \nu_1)} + \sum_{j=1}^n \|F_j u_m - F_j u\|_{L^{p'}(\Omega, \omega_2)} \right. \\
 &\quad \left. + C_{2,1} \|H u_m - H u\|_{L^{s'}(\Omega, \nu_2)} + \|J u_m - J u\|_{L^{p'}(\Omega, \omega_1)} \right) \|(f, \nabla f)\|_X.
 \end{aligned}$$

Hence, using (3.15), (3.16), (3.17) and (3.18), we have

$$\langle \mathcal{T}(u_m) - \mathcal{T}(u), (f, \nabla f) \rangle \rightarrow 0$$

as $m \rightarrow \infty$, that is, $\mathcal{T}$ is weakly continuous. $\square$

4. Proof of Theorem 1.1.

Existence of solution. Based on Lemma 3.1, Lemma 3.2 and Lemma 3.3, by Theorem 2.4 there exists an element $(u, \nabla u) \in \mathcal{K}_\psi$ such that

$$\langle T(u), (f, \nabla f) - (u, \nabla u) \rangle \geq 0$$

whenever $(f, \nabla f) \in \mathcal{K}_\psi$. For each $\varphi \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$, we have $u + \varphi - \psi = (u - \psi) + \varphi \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$. Therefore, $(u + \varphi, \nabla u + \nabla \varphi) \in \mathcal{K}_\psi$ and $(u - \varphi, \nabla u - \nabla \varphi) \in \mathcal{K}_\psi$. Then

$$\langle \mathcal{T}(u), (\varphi, \nabla \varphi) \rangle = \langle \mathcal{T}(u), (u + \varphi, \nabla u + \nabla \varphi) - (u, \nabla u) \rangle \geq 0$$

and

$$\langle \mathcal{T}(u), (\varphi, \nabla \varphi) \rangle = -\langle \mathcal{T}(u), (u - \varphi, \nabla u - \nabla \varphi) - (u, \nabla u) \rangle \leq 0.$$

Therefore, $\langle \mathcal{T}(u), (\varphi, \nabla \varphi) \rangle = 0$, that is,

$$\begin{aligned} & \int_{\Omega} \mathcal{B}(x, \nabla u) \cdot \nabla \varphi \nu_1 dx + \int_{\Omega} \mathcal{A}(x, \nabla u) \cdot \nabla \varphi \omega_2 dx \\ & + \int_{\Omega} \mathcal{H}(x, u, \nabla u) \varphi \nu_2 dx + \int_{\Omega} |u|^{p-2} u \varphi \omega_1 dx \\ & = \int_{\Omega} \rho_0 \varphi dx + \sum_{j=1}^n \int_{\Omega} \rho_j D_j \varphi dx, \end{aligned}$$

for all $\varphi \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$, that is, u is a solution to problem (P).

Uniqueness. For the uniqueness, let u_1 and u_2 be two solutions to problem (P), with $u_i - \psi \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$ ($i = 1, 2$). Since $\psi, u_1, u_2 \in W^{1,p}(\Omega, \omega_1, \omega_2)$ and $u_1 - u_2 = (u_1 - \psi) - (u_2 - \psi) \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$, then

$$\begin{aligned} & \int_{\Omega} \mathcal{B}(x, \nabla u_1) \cdot \nabla (u_1 - u_2) \nu_1 dx + \int_{\Omega} \mathcal{A}(x, \nabla u_1) \cdot \nabla (u_1 - u_2) \omega_2 dx \\ & + \int_{\Omega} \mathcal{H}(x, u_1, \nabla u_1) (u_1 - u_2) \nu_2 dx + \int_{\Omega} |u_1|^{p-2} u_1 (u_1 - u_2) \omega_1 dx \\ & = \int_{\Omega} \rho_0 (u_1 - u_2) dx + \int_{\Omega} \rho \cdot \nabla (u_1 - u_2) dx, \end{aligned}$$

and

$$\begin{aligned} & \int_{\Omega} \mathcal{B}(x, \nabla u_2) \cdot \nabla (u_1 - u_2) \nu_1 dx + \int_{\Omega} \mathcal{A}(x, \nabla u_2) \cdot \nabla (u_1 - u_2) \omega_2 dx \\ & + \int_{\Omega} \mathcal{H}(x, u_2, \nabla u_2) (u_1 - u_2) \nu_2 dx + \int_{\Omega} |u_2|^{p-2} u_2 (u_1 - u_2) \omega_1 dx \\ & = \int_{\Omega} \rho_0 (u_1 - u_2) dx + \int_{\Omega} \rho \cdot \nabla (u_1 - u_2) dx. \end{aligned}$$

Hence,

$$\begin{aligned} & \int_{\Omega} \left(\mathcal{B}(x, \nabla u_1) - \mathcal{B}(x, \nabla u_2) \right) \cdot \nabla(u_1 - u_2) \nu_1 dx \\ & + \int_{\Omega} \left(\mathcal{A}(x, \nabla u_1) - \mathcal{A}(x, \nabla u_2) \right) \cdot \nabla(u_1 - u_2) \omega_2 dx \\ & + \int_{\Omega} (\mathcal{H}(x, u_1, \nabla u_1) - \mathcal{H}(x, u_2, \nabla u_2))(u_1 - u_2) \nu_2 dx \\ & + \int_{\Omega} (|u_1|^{p-2}u_1 - |u_2|^{p-2}u_2)(u_1 - u_2) \omega_1 dx = 0, \end{aligned}$$

and by (H2), (H6), (H10) and Proposition 2.3 we have

$$\begin{aligned} & \int_{\Omega} \left(\mathcal{B}(x, \nabla u_1) - \mathcal{B}(x, \nabla u_2) \right) \cdot \nabla(u_1 - u_2) \nu_1 dx = 0 \\ & \int_{\Omega} \left(\mathcal{A}(x, \nabla u_1) - \mathcal{A}(x, \nabla u_2) \right) \cdot \nabla(u_1 - u_2) \omega_2 dx = 0, \\ & \int_{\Omega} \left(\mathcal{H}(x, u_1, \nabla u_1) - \mathcal{H}(x, u_2, \nabla u_2) \right) (u_1 - u_2) \nu_2 dx = 0, \\ & \int_{\Omega} (|u_1|^{p-2}u_1 - |u_2|^{p-2}u_2)(u_1 - u_2) \omega_1 dx = 0. \end{aligned}$$

Thus,

$$\int_{\{\nabla u_1 \neq \nabla u_2\}} \left(\mathcal{A}(x, \nabla u_1) - \mathcal{A}(x, \nabla u_2) \right) \cdot \nabla(u_1 - u_2) \omega_2 dx = 0.$$

By (H2) and $\omega_2 \in A_p$, we obtain $|\{\nabla u_1 \neq \nabla u_2\}| = 0$ and $\nabla u_1 = \nabla u_2$ a.e. in Ω . Since $u_1 - u_2 \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$, we conclude that $u_1 = u_2$ a.e. in Ω . Theorem 1.1 is thereby proved.

Example 4.1. Let $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. For $(x, y) \in \mathbb{R}^2$, $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, $\eta \in \mathbb{R}$, $p = 4$ and $q = s = 3$, consider the functions

$$\begin{aligned} \mathcal{A}((x, y), \xi) &= \xi |\xi|^2 (\cos(x/(x^2 + y^2)) + 2) \\ \mathcal{B}((x, y), \xi) &= \xi |\xi| (\sin(y/(x^2 + y^2)) + 2), \\ \mathcal{H}((x, y), \eta, \xi) &= \eta |\eta| (1 + \cos^2(xy)) \\ \omega_1(x, y) &= (x^2 + y^2)^{-1/2} \quad \text{and} \quad \omega_2(x, y) = (x^2 + y^2)^{1/2} \\ \nu_1(x, y) &= (x^2 + y^2)^2 \quad \text{and} \quad \nu_2(x, y) = (x^2 + y^2)^{-1/8}, \\ \rho_0(x, y) &= \frac{\cos(xy)}{(x^2 + y^2)} \quad \text{and} \quad \rho(x, y) = \left(\frac{\cos(xy)}{x^2 + y^2}, \frac{\sin(xy)}{x^2 + y^2} \right), \\ \text{and } \psi(x, y) &= \sin(x + y). \end{aligned}$$

Therefore, by Theorem 1.1, the problem (P) has a unique solution $u \in W^{1,4}(\Omega, \omega_1, \omega_2)$.

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