doi: 10.17951/a.2025.79.1.1-12

ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. LXXIX, NO. 1, 2025 SECTIO A 1–12

MIROSLAV DOUPOVEC, JAN KUREK
and WŁODZIMIERZ M. MIKULSKI

The Legendre-like operators on tuples of Lagrangians and functions

Abstract. Let Y be a fibred manifold with an m-dimensional basis M. We describe all Legendre-like operators C, i.e. natural operators transforming tuples (λ, g) of Lagrangians $\lambda : J^sY \to \bigwedge^m T^*M$ and functions $g : M \to \mathbf{R}$ (resp. $g : Y \to \mathbf{R}$) into Legendre maps $C(\lambda, g) : J^sY \to S^sTM \otimes V^*Y \otimes \bigwedge^m T^*M$. The most important example of such operators is the Legendre operator (from the variational calculus) being the one in question depending only on Lagrangians.

1. Introduction. All manifolds and maps between manifolds considered in this paper are assumed to be smooth (i.e. of class $\mathcal{C}^\infty$).

Let $\mathcal{FM}_{m,n}$ denote the category of fibred manifolds with m-dimensional bases and n-dimensional fibres and their fibred diffeomorphisms onto open images.

Given an $\mathcal{FM}_{m,n}$-object $Y \to M$, we have the s-jet prolongation J^sY of $Y \to M$ for any positive integer s. Thus J^sY is the space of all s-jets $j^s_x\sigma$ at $x \in M$ of local sections $\sigma : M \to Y$ of $Y \to M$. If $f : Y \to Y^1$ is an $\mathcal{FM}_{m,n}$-map with the base map $\underline{f} : M \to M_1$, then we have the induced fibred map $J^sf : J^sY \to J^sY_1$ given by $J^rf(j^s_{x_o}\sigma) = j^s_{\underline{f}(x_o)}(f \circ \sigma \circ \underline{f}^{-1})$,

2010 *Mathematics Subject Classification.* 58A20, 58E30.

Key words and phrases. Fibred manifolds, Lagrangians, Legendre maps, natural operators, Legendre transformation.

$j^s_{x_o}\sigma \in J^s_{x_o}Y$, $x_o \in M$. So, we have the bundle functor $J^r : \mathcal{FM}_{m,n} \to \mathcal{FM}$ in the sense of [2].

Given an $\mathcal{FM}_{m,n}$-object $Y \to M$, we also have the vertical bundle $VY \to Y$ and its dual bundle $V^*Y \to Y$ and the cotangent bundle T^*M and its mth inner product $\bigwedge^m T^*M$ and the tangent bundle TM and its sth symmetric product S^sTM.

Given fibred manifolds $Z_1 \to M$ and $Z_2 \to M$ with the same basis M, let $\mathcal{C}^\infty_M(Z_1, Z_2)$ denote the space of all base preserving fibred maps of Z_1 into Z_2. Elements from the space $\mathcal{C}^\infty_M(J^sY, \bigwedge^m T^*M)$ are called (sth order) Lagrangians on $Y \to M$. Elements from the space $\mathcal{C}^\infty_Y(J^sY, S^sTM \otimes V^*Y \otimes \bigwedge^m T^*M)$ are called Legendre maps on $Y \to M$.

Any sth order Lagrangian $\lambda : J^sY \to \bigwedge^m T^*M$ on an $\mathcal{FM}_{m,n}$-object $Y \to M$ induces canonically the Legendre transformation $\Lambda(\lambda) : J^sY \to S^sTM \otimes V^*Y \otimes \bigwedge^m T^*M$, see e.g. Example 2.1 below. So, we have the $\mathcal{FM}_{m,n}$-natural operator

$$\Lambda : \mathcal{C}^\infty_M\left(J^sY, \bigwedge^m T^*M\right) \to \mathcal{C}^\infty_Y\left(J^sY, S^sTM \otimes V^*Y \otimes \bigwedge^m T^*M\right).$$

It is called the Legendre operator. We inform that the concept of natural operators can be found in [2].

The Legendre transformation $\Lambda(\lambda)$ plays an important role in analytical mechanics, especially in the case of regular Lagrangians λ, the transformation $\Lambda(\lambda)$ can be considered as the corresponding $J^{s-1}Y$-preserving diffeomorphism between J^sY and $(\pi^{s-1}_0)^*(S^sTM \otimes V^*Y \otimes \bigwedge^m T^*M)$ (then it joints the Lagrange and Hamilton formalisms in fibred manifolds), see [1].

In [5], it is proved that given positive integers m, n and s, any local $\mathcal{FM}_{m,n}$-natural regular operator $\mathcal{C}^\infty_M(J^sY, \bigwedge^m T^*M) \to \mathcal{C}^\infty_Y(J^sY, S^sTM \otimes V^*Y \otimes \bigwedge^m T^*M)$ is of the form $c\Lambda$, $c \in \mathbf{R}$, where Λ is the Legendre operator.

In the present paper, if $m \geq 3$, we describe all Legendre-like operators, i.e. local $\mathcal{FM}_{m,n}$-natural regular operators

$$C : \mathcal{C}^\infty_M\left(J^sY, \bigwedge^m T^*M\right) \times \mathcal{C}^\infty(M, \mathbf{R}) \to \mathcal{C}^\infty_Y\left(J^sY, S^sTM \otimes V^*Y \otimes \bigwedge^m T^*M\right)$$

(resp. $C : \mathcal{C}^\infty_M\left(J^sY, \bigwedge^m T^*M\right) \times \mathcal{C}^\infty(Y, \mathbf{R}) \to \mathcal{C}^\infty_Y\left(J^sY, S^sTM \otimes V^*Y \otimes \bigwedge^m T^*M\right)$) transforming a tuple (λ, g) of a Lagrangian $\lambda \in \mathcal{C}^\infty_M(J^sY, \bigwedge^m T^*M)$ on an $\mathcal{FM}_{m,n}$-object $Y \to M$ and a real valued map $g \in \mathcal{C}^\infty(M, \mathbf{R})$ on the base M of $Y \to M$ (resp. $g \in \mathcal{C}^\infty(Y, \mathbf{R})$ on the total space Y of $Y \to M$) into a Legendre map $C(\lambda, g) \in \mathcal{C}^\infty_Y(J^sY, S^sTM \otimes V^*Y \otimes \bigwedge^m T^*M)$.

The $\mathcal{FM}_{m,n}$-naturality (or invariance) of C means that for any $\mathcal{FM}_{m,n}$-map $f : Y \to Y_1$ with the base map $\underline{f} : M \to M_1$ and Lagrangians $\lambda \in \mathcal{C}^\infty_M(J^sY, \bigwedge^m T^*M)$ and $\lambda_1 \in \mathcal{C}^\infty_{M_1}(J^sY_1, \bigwedge^m TM_1)$ and maps $g : M \to \mathbf{R}$ and $g_1 : M_1 \to \mathbf{R}$ (resp. $g : Y \to \mathbf{R}$ and $g_1 : Y_1 \to \mathbf{R}$), if λ and λ_1 are f-related and g and g_1 are f-related, then so are $C(\lambda, g)$ and $C(\lambda_1, g_1)$. The locality of C means that $C(\lambda, g)_\rho$ depends on $\mathrm{germ}_\rho(\lambda, g)$ for any $\rho \in J^sY$ and $\lambda \in \mathcal{C}^\infty_M(J^sY, \bigwedge^m T^*M)$ and any $g \in \mathcal{C}^\infty(M, \mathbf{R})$ (resp. $g \in \mathcal{C}^\infty(Y, \mathbf{R})$). The regularity means that C sends smoothly parametrized families of tuples of Lagrangians and maps into smoothly parametrized families of Legendre maps.

The present paper is a continuation of [3], where using a similar procedure, we described all Euler-like operators, i.e. local $\mathcal{FM}_{m,n}$-natural regular operators

$$\tilde{C} : \mathcal{C}^\infty_M\left(J^sY, \bigwedge^m T^*M\right) \times \mathcal{C}^\infty(M, \mathbf{R}) \to \mathcal{C}^\infty_Y\left(J^{2s}Y, V^*Y \otimes \bigwedge^m T^*M\right).$$

2. The Legendre-like operators on tuples of Lagrangians and functions on bases.

Example 2.1. Let $\lambda : J^sY \to \bigwedge^m T^*M$ be an s-order Lagrangian on an $\mathcal{FM}_{m,n}$-object $Y \to M$. Let $\delta\lambda : \mathcal{C}^\infty_{J^sY}(J^sY, V^*J^sY \otimes \bigwedge^m T^*M)$ be the vertical differential of λ, i.e. the composition of the restriction $\tilde{\delta}\lambda : VJ^sY \to V\bigwedge^m T^*M = \bigwedge^m T^*M \times_M \bigwedge^m T^*M$ of the differential $d\lambda : TJ^sY \to T\bigwedge^m T^*M$ of λ to the vertical sub-bundles with the second (essential) factor projection $\bigwedge^m T^*M \times_M \bigwedge^m T^*M \to \bigwedge^m T^*M$. Let $\Lambda(\lambda) : S^sT^*M \otimes VY \to \bigwedge^m T^*M$ be the restriction of $\delta\lambda : VJ^sY \to \bigwedge^m T^*M$ to the vector-subbundle $S^sT^*M \otimes VY \subset VJ^sY$, the kernel of $V\pi^s_{s-1} : VJ^sY \to VJ^{s-1}Y$, where $\pi^s_{s-1} : J^sY \to J^{s-1}Y$ is the jet projection. The map $\Lambda(\lambda)$ is called the Legendre transformation. So, we have the $\mathcal{FM}_{m,n}$-natural operator

$$\Lambda : \mathcal{C}^\infty_M\left(J^sY, \bigwedge^m T^*M\right) \to \mathcal{C}^\infty_Y\left(J^sY, S^sTM \otimes V^*Y \otimes \bigwedge^m T^*M\right).$$

The natural operator Λ is called the Legendre operator.

We have the following:

Theorem 2.2. *Let m, n, s be positive integers. If $m \geq 3$, then any local $\mathcal{FM}_{m,n}$-natural regular operator*

$$C : \mathcal{C}^\infty_M\left(J^sY, \bigwedge^m T^*M\right) \times \mathcal{C}^\infty(M, \mathbf{R}) \to \mathcal{C}^\infty_Y\left(J^sY, S^sTM \otimes V^*Y \otimes \bigwedge^m T^*M\right)$$

is $h \cdot \Lambda$ for a (uniquely determined by C) map $h : \mathbf{R} \to \mathbf{R}$, where $h \cdot C$ is defined by

$$(h \cdot C)(\lambda, g)_{|j^s_{x_o}\sigma} = h(g(x_o)) \cdot C(\lambda, g)_{|j^s_{x_o}\sigma}$$

for any $h : \mathbf{R} \to \mathbf{R}$ and any C in question and any $\lambda, g, j^s_{x_o}\sigma$ as above and where Λ is the Legendre operator.

So, the space of all C (as in Theorem 2.2) is the free 1-dimensional $\mathcal{C}^\infty(\mathbf{R})$-module and the operator Λ forms the basis in this module.

The proof of Theorem 2.2 will be given in Section 4.

From Theorem 2.2 it follows the following result of [5]:

Corollary 2.3. *Let m,n,s be positive integers. If $m\geq 3$, then any local $\mathcal{FM}_{m,n}$-natural regular operator*

$$C:\mathcal{C}^\infty_M\left(J^sY,\bigwedge^m T^*M\right)\to \mathcal{C}^\infty_Y\left(J^sY,S^sTM\otimes V^*Y\otimes \bigwedge^m T^*M\right)$$

is $a\Lambda$ for some (uniquely determined by C) real number a, where Λ is the Legendre operator.

Proof. Let C be an operator in question. By Theorem 2.2, we can write $C=h\cdot\Lambda$ for some (uniquely determined by C) map $h:\mathbf{R}\to\mathbf{R}$. Then $C(\lambda)=C(\lambda,1)=h(1)\Lambda(\lambda)$, i.e. $C=h(1)\Lambda$. □

In [5], the corollary is proved for $m\in\{1,2\}$, too.

3. Some transformation rules. Let $\mathbf{N}$ be the set of non-negative integers and let $\mathbf{R}^{m,n}$ be the trivial $\mathcal{FM}_{m,n}$-object $\mathbf{R}^m\times\mathbf{R}^n\to\mathbf{R}^m$ and let $x^1,\dots,x^m,y^1,\dots,y^n$ be the usual coordinates on $\mathbf{R}^{m,n}$. Given $i=1,\dots,m$ let $1_i:=(0,\dots,0,1,0,\dots,0)\in\mathbf{N}^m$, where 1 occupies ith position.

We have the induced coordinates $((x^i),(y^j_\alpha))$ on $J^s(\mathbf{R}^{m,n})$, where $i=1,\dots,m$ and $j=1,\dots,n$ and $\alpha=(\alpha_1,\dots\alpha_m)\in\mathbf{N}^m$ are such that $|\alpha|=\alpha_1+\dots+\alpha_m\leq s$. They are given by

$$x^i(j^s_{x_o}\sigma)=x^i_o \quad\text{and}\quad y^j_\alpha(j^s_{x_o}\sigma)=(\partial_\alpha\sigma^j)(x_o)$$

for any $j^s_{x_o}\sigma=j^s_{x_o}(\sigma^1,\dots,\sigma^n)\in J^s_{x_o}(\mathbf{R}^{m,n})=J^s_{x_o}(\mathbf{R}^m,\mathbf{R}^n)$, $x_o\in\mathbf{R}^m$, where ∂_α is the iterated partial derivative as indicated multiplied by $\frac{1}{\alpha!}$.

Lemma 3.1. *Let $i=1,\dots,m$ and $j=1,\dots,n$ and $\alpha=(\alpha_1,\dots,\alpha_m)\in\mathbf{N}^m$ be such that $|\alpha|\leq s$.*

(i) For any $\tau=(\tau^1,\dots,\tau^n)\in(\mathbf{R}\setminus\{0\})^n$, we have

$$(J^s\psi_\tau)_*y^j_\alpha=\tau^j y^j_\alpha\,,$$

where $\psi_\tau=(x^1,\dots,x^m,\frac{1}{\tau^1}y^1,\dots,\frac{1}{\tau^n}y^n)$ is the $\mathcal{FM}_{m,n}$-map.

(ii) For any $t\in\mathbf{R}\setminus\{0\}$, we have

$$(J^s\varphi^i_t)_*y^j_\alpha=t^{-\alpha_i}y^j_\alpha\,,$$

where $\varphi^i_t=(x^1,\dots,\frac{1}{t}x^i,\dots,x^m,y^1,\dots,y^n)$ is the $\mathcal{FM}_{m,n}$-map.

Proof. We prove (i), only. We have

$$\begin{aligned}((J^s\psi_\tau)_*y^j_\alpha)(j^s_{x_o}\sigma)&=y^j_\alpha(J^s\psi^{-1}_\tau(j^s_{x_o}\sigma))=y^j_\alpha(j^s_{x_o}(\psi^{-1}_\tau\circ\sigma))\\&=\partial_\alpha(\tau^j\sigma^j)(x_o)=\tau^j\partial_\alpha(\sigma^j)(x_o)=\tau^j y^j_\alpha(j^s_{x_o}\sigma)\,.\end{aligned}$$

The proof of (ii) is quite similar. □

4. Proof of Theorem 2.2. We will use the notations as in the previous sections. Additionally, let $dx^\mu := dx^1 \wedge \dots \wedge dx^m$ and let $x^\alpha := (x^1)^{\alpha_1} \cdot \ldots \cdot (x^m)^{\alpha_m}$ for any $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbf{N}^m$.

Consider a local $\mathcal{FM}_{m,n}$-natural regular operator

$$C : \mathcal{C}^\infty_M\left(J^sY, \bigwedge^m T^*M\right) \times \mathcal{C}^\infty(M, \mathbf{R}) \to \mathcal{C}^\infty_Y\left(J^sY, S^sTM \otimes V^*Y \otimes \bigwedge^m T^*M\right).$$

Assume $m \geq 3$. We prove several lemmas.

Lemma 4.1. *Our operator C is determined by the values*

$$\left\langle C(\lambda, g)_\rho, \bigodot^s d_0\omega \otimes v \right\rangle \in \bigwedge^m T^*_0\mathbf{R}^m \tag{1}$$

for all $\lambda \in \mathcal{C}^\infty_{\mathbf{R}^m}(J^s(\mathbf{R}^{m,n}), \bigwedge^m T^*\mathbf{R}^m)$, all $v \in T_0\mathbf{R}^n = V_{(0,0)}\mathbf{R}^{m,n}$, all $d_0\omega \in T^*_0\mathbf{R}^m$, all $\rho = j^s_0(\sigma) \in J^s_0(\mathbf{R}^m, \mathbf{R}^n)_0 = J^s_{(0,0)}(\mathbf{R}^{m,n})$ and all $g : \mathbf{R}^m \to \mathbf{R}$, where $\bigodot^s d_0\omega = d_0\omega \bigodot \cdots \bigodot d_0\omega$ (s times of $d_0\omega$). (In other words, if C' is another such operator giving the same as C collection of values (1), then $C = C'$.)

Proof. It follows immediately from the invariance of C with respect to $\mathcal{FM}_{m,n}$-charts. □

Lemma 4.2. *Our operator C is determined by the values*

$$\left\langle C(\lambda, g)_\theta, \bigodot^s d_0\omega \otimes v \right\rangle \in \bigwedge^m T^*_0\mathbf{R}^m \tag{2}$$

for all $\lambda \in \mathcal{C}^\infty_{\mathbf{R}^m}(J^s(\mathbf{R}^{m,n}), \bigwedge^m T^*\mathbf{R}^m)$, all $v \in T_0\mathbf{R}^n = V_{(0,0)}\mathbf{R}^{m,n}$, all $d_0\omega \in T^*_0\mathbf{R}^m$ and all $g : \mathbf{R}^m \to \mathbf{R}$, where $\theta := j^s_0(0) \in J^s_0(\mathbf{R}^m, \mathbf{R}^n)_0 = J^s_{(0,0)}(\mathbf{R}^{m,n})$.

Proof. Given a map σ with $\rho = j^s_0(\sigma) \in J^s_0(\mathbf{R}^m, \mathbf{R}^n)_0 = J^s_{(0,0)}(\mathbf{R}^{m,n})$, we have $\nu : \mathbf{R}^{m,n} \to \mathbf{R}^{m,n}$ defined by $\nu := (x, y - \sigma(x))$, where $x = (x^1, \dots, x^m)$ and $y = (y^1, \dots, y^n)$. This $\mathcal{FM}_{m,n}$-map ν transforms $j^s_0(\sigma)$ into θ. Then using the previous lemma and the $\mathcal{FM}_{m,n}$-invariance of C with respect to ν, we end the proof of the lemma. □

Lemma 4.3. *Our operator C is determined by the values*

$$\left\langle C(\lambda, x^m + c)_\theta, \bigodot^s d_0\omega \otimes v \right\rangle \in \bigwedge^m T^*_0\mathbf{R}^m \tag{3}$$

for all $\lambda \in \mathcal{C}^\infty_{\mathbf{R}^m}(J^s(\mathbf{R}^{m,n}), \bigwedge^m T^*\mathbf{R}^m)$, all $v \in T_0\mathbf{R}^n = V_{(0,0)}\mathbf{R}^{m,n}$, all $d_0\omega \in T^*_0\mathbf{R}^m$ and all $c \in \mathbf{R}$, where θ is as above.

Proof. Because of the regularity of C, we have additional assume $d_0 g \neq 0$. Then using the previous lemma and the invariance of C with respect to the $(0,0)$-preserving $\mathcal{FM}_{m,n}$-map

$$(x^1, \dots, x^{m-1}, g(x^1, \dots, x^m) - g(0, \dots, 0), y^1, \dots, y^n)$$

(it preserves θ and transforms g into $x^m + c$) with $c = g(0)$, we end the proof of the lemma. □

Lemma 4.4. *Our operator C is determined by the values*

$$\left\langle C(\lambda, x^m + c)_\theta, \bigodot^s d_0\omega \otimes v_o \right\rangle \in \bigwedge^m T_0^*\mathbf{R}^m \tag{4}$$

for all $\lambda \in \mathcal{C}^\infty_{\mathbf{R}^m}(J^s(\mathbf{R}^{m,n}), \bigwedge^m T^*\mathbf{R}^m)$, all $d_0\omega \in T_0^*\mathbf{R}^m$ and all $c \in \mathbf{R}$, where θ is as above and $v_o := \frac{\partial}{\partial y^1}_{|(0,0)} \in T_0\mathbf{R}^n = V_{(0,0)}\mathbf{R}^{m,n}$.

Proof. In the previous lemma, we can obviously additionally assume that $v \neq 0$. Then there exists an $\mathcal{FM}_{m,n}$-map Φ of the form $\mathrm{id}_{\mathbf{R}^m} \times \phi$, where $\phi : \mathbf{R}^n \to \mathbf{R}^n$ is a linear isomorphism, sending v into $v_o = \frac{\partial}{\partial y^1}_{|(0,0)}$. Such Φ preserves θ and $x^m + c$. Then using the previous lemma and the invariance of C with respect to Φ , we complete the proof of the lemma. □

Lemma 4.5. *Our operator C is determined by the values*

$$\left\langle C(L((x^i), (y^j_\alpha))dx^\mu + bdx^\mu, x^m + c)_\theta, \bigodot^s d_0\omega \otimes v_o \right\rangle \in \bigwedge^m T_0^*\mathbf{R}^m \tag{5}$$

for all $L : \mathbf{R}^{m,n} \to \mathbf{R}$ with $L((x^i), (0)) = 0$, all $b, c \in \mathbf{R}$ and all $d_0\omega \in T_0^*\mathbf{R}^m$, where θ and v_o are as above.

Proof. In the previous lemma, we can write

$$\lambda = L((x^i), (y^j_\alpha))dx^\mu + f(x^1, \ldots, x^m)dx^\mu ,$$

where L and f are arbitrary real valued maps with $L((x^i), (0)) = 0$. By the regularity of C, we can assume $f(0) \neq 0$. Then, using the previous lemma and the invariance of C with respect to the $\mathcal{FM}_{m,n}$-map

$$\Psi = (F(x^1, .., x^m), x^2, \ldots, x^m, y^1, \ldots, y^n)^{-1} ,$$

where $\frac{\partial}{\partial x^1}F = f$ and $F(0, x^2, \ldots, x^m) = 0$, we may additionally assume $f = 1$ because Ψ preserves θ and $g = x^m + c$ and $v_o = \frac{\partial}{\partial y^1}_{|(0,0)}$ and it sends dx^μ into $f dx^\mu$. The proof of the lemma is complete. □

Lemma 4.6. *Our operator C is determined by the values*

$$\left\langle C(L((x^i), (y^j_\alpha))dx^\mu + bdx^\mu, x^m + c)_\theta, \bigodot^s d_0\omega_o \otimes v_o \right\rangle \in \bigwedge^m T_0^*\mathbf{R}^m \tag{6}$$

for all $L : \mathbf{R}^{m,n} \to \mathbf{R}$ with $L((x^i), (0)) = 0$ and all $b, c \in \mathbf{R}$, where θ and v_o are as above and $d_0\omega_o := d_0x^{m-1}$.

Proof. In the previous lemma, because of the regularity of C, we can assume that $d_0(x^m + c)$ and $d_0\omega$ are linearly independent. Then by the previous lemma and the invariance of C with respect to an $\mathcal{FM}_{m,n}$-map of the form $(\varphi(x), y^1, \ldots, y^n)$ with linear φ (it preserves θ and v_o and it sends dx^μ into $det(\varphi) \cdot dx^\mu$) we end the proof of the lemma. □

Lemma 4.7. *Our operator C is determined by the values*

$$\left\langle C(ax^\beta y^1_\alpha dx^\mu + bdx^\mu, x^m + c)_\theta, \bigodot^s d_0x^{m-1} \otimes \frac{\partial}{\partial y^1}_{|(0,0)}\right\rangle \tag{7}$$

for all $\alpha,\beta \in \mathbf{N}^m$ with $|\alpha| \leq s$ and all $a,b,c \in \mathbf{R}$. Moreover given b and c, the value (7) *depends linearly on a.*

Proof. Because of the locality of C, using the main result of [4], we may additionally assume that in the previous lemma the maps L are arbitrary polynomials in $((x^i),(y^j_\alpha))$ of degree $\leq q$ with $L((x^i),(0)) = 0$, where q is an arbitrary positive integer.

Further, by the invariance of C with respect to $(x^1,\dots,x^m,\frac{1}{\tau^1}y^1,\dots,\frac{1}{\tau^n}y^n)$ being $\mathcal{FM}_{m,n}$-map for any $(\tau^1,\dots,\tau^n) \in (\mathbf{R}\setminus\{0\})^n$, we get the homogeneity condition

$$\left\langle C(L((x^i),(\tau^j y^j_\alpha))dx^\mu + bdx^\mu, x^m + c)_\theta, \bigodot^s d_0x^{m-1} \otimes \frac{\partial}{\partial y^1}_{|(0,0)}\right\rangle$$
$$= \tau^1 \left\langle C(L((x^i),(y^j_\alpha))dx^\mu + bdx^\mu, x^m + c)_\theta, \bigodot^s d_0x^{m-1} \otimes \frac{\partial}{\partial y^1}_{|(0,0)}\right\rangle,$$

see Lemma 3.1 (i). Then by the homogeneous function theorem (see [2]), given b and c, the value $\left\langle C(Ldx^\mu + bdx^\mu, x^m + c)_\theta, \bigodot^s d_0x^{m-1} \otimes \frac{\partial}{\partial y^1}_{|(0,0)}\right\rangle$ depends linearly on the coefficients of L on $x^\beta y^1_\alpha$ and it is independent of the other coefficients of L. Now, because of Lemma 4.6, our lemma is clear. □

Lemma 4.8. *Our operator C is determined by the values*

$$\left\langle C(ax^\beta y^1_\alpha dx^\mu + bdx^\mu, ex^m + c)_\theta, \bigodot^s d_0x^{m-1} \otimes \frac{\partial}{\partial y^1}_{|(0,0)}\right\rangle \tag{8}$$

for all $\alpha,\beta \in \mathbf{N}^m$ with $|\alpha| \leq s$ and all $a,b,c,e \in \mathbf{R}$. Moreover given b and c and e, the value (8) *depends linearly on a.*

Proof. In the proof of the previous lemma it suffices to replace x^m by ex^m. □

Lemma 4.9. *We have*

$$\left\langle C(ax^\beta y^1_\alpha dx^\mu + bdx^\mu, x^m + c)_\theta, \bigodot^s d_0x^{m-1} \otimes \frac{\partial}{\partial y^1}_{|(0,0)}\right\rangle = 0 \tag{9}$$

for any $\beta,\alpha \in \mathbf{N}^m$ with both $|\alpha| \leq s$ and ($\beta_1 > \alpha_1$ or ... or $\beta_{m-2} > \alpha_{m-2}$ or $\beta_{m-1} > \alpha_{m-1} - s$ or $\beta_m > \alpha_m$).

Proof. By the invariance of C with respect to $\varphi^i_t = (x^1,\dots,\frac{1}{t}x^i,\dots,x^m,y^1,\dots,y^n)$ being $\mathcal{FM}_{m,n}$-map for any $t \in \mathbf{R}\setminus\{0\}$ and any $i = 1,\dots,m$ and using the fact that given b and c and e the values (8) depend linearly on a, we get the condition

$$t^{\beta_i-\alpha_i+\delta^{i,m-1}s}\left\langle C(ax^\beta y^1_\alpha dx^\mu + tbdx^\mu, t^{\delta_{im}}x^m + c)_\theta, \bigodot^s d_0x^{m-1}\otimes \frac{\partial}{\partial y^1}_{|(0,0)}\right\rangle$$
$$= \left\langle C(ax^\beta y^1_\alpha dx^\mu + bdx^\mu, x^m + c)_\theta, \bigodot^s d_0x^{m-1}\otimes \frac{\partial}{\partial y^1}_{|(0,0)}\right\rangle$$

because φ^i_t preserves C and θ and $\frac{\partial}{\partial y^1}_{|(0,0)}$ and it sends x^β into $t^{\beta_i}x^\beta$ and it sends x^m into $t^{\delta_{i,m}}x^m$ (the Kronecker delta) and it sends x^{m-1} into $t^{\delta_{i,m-1}}x^{m-1}$ and it sends y^1_α into $t^{-\alpha_i}y^1_\alpha$ and it sends dx^μ into tdx^μ, see Lemma 3.1 (ii). Then putting $t\to 0$, we end the proof of the lemma. □

Lemma 4.10. *Our operator C is determined by the values* (7) *for all $a,b,c\in\mathbf{R}$ and all $\alpha,\beta\in\mathbf{N}^m$ with ($|\alpha|\le s$ and $\beta_1\le\alpha_1$ and ... and $\beta_{m-2}\le\alpha_{m-2}$ and $\beta_{m-1}\le\alpha_{m-1}-s$ and $\beta_m\le\alpha_m$). Moreover given b and c, the value* (7) *depends linearly on a.*

Proof. It follows from Lemmas 4.9 and 4.7. □

Lemma 4.11. *Our operator C is determined by the values*

$$\left\langle C(ay^1_{(0,\dots,0,s,0)}dx^\mu + bdx^\mu, x^m + c)_\theta, \bigodot^s d_0x^{m-1}\otimes \frac{\partial}{\partial y^1}_{|(0,0)}\right\rangle \in \bigwedge^m T^*_0\mathbf{R}^m$$

for all $a,b,c\in\mathbf{R}$. Moreover, given b and c, the above values depend linearly on a.

Proof. It is an immediate consequence of the previous lemma. □

Lemma 4.12. *Our operator C is determined by the values*

$$\left\langle C(y^1_{(0,\dots,0,s,0)}dx^\mu, x^m + c)_\theta, \bigodot^s d_0x^{m-1}\otimes \frac{\partial}{\partial y^1}_{|(0,0)}\right\rangle \in \bigwedge^m T^*_0\mathbf{R}^m$$

for all $c\in\mathbf{R}$.

Proof. Using the invariance of C with respect to $(\frac{1}{t}x^1, x^2,\dots,x^m,y^1,\dots,y^n)$, we get the condition

$$\left\langle C(ay^1_{(0,\dots,0,s,0)}dx^\mu + tbdx^\mu, x^m + c)_\theta, \bigodot^s d_0x^{m-1}\otimes \frac{\partial}{\partial y^1}_{|(0,0)}\right\rangle$$
$$= \left\langle C(ay^1_{(0,\dots,0,s,0)}dx^\mu + bdx^\mu, x^m + c)_\theta, \bigodot^s d_0x^{m-1}\otimes \frac{\partial}{\partial y^1}_{|(0,0)}\right\rangle.$$

(Here we use $m\ge 3$.) Putting $t\to 0$, we see that

$$\left\langle C(ay^1_{(0,\dots,0,s,0)}dx^\mu + bdx^\mu, x^m + c)_\theta, \bigodot^s d_0x^{m-1}\otimes \frac{\partial}{\partial y^1}_{|(0,0)}\right\rangle$$

is independent of b. Then applying the previous lemma, we complete the proof. $\square$

Now, we are in position to prove Theorem 2.2.

Proof. Because of Lemma 4.12, our operator C is determined by the map $C^{<o>}:\mathbf{R}\to\mathbf{R}$ defined by

$$C^{<o>}(c)dx^\mu_{|0} := \left\langle C(y^1_{(0,\dots,s,0)}dx^\mu, x^m+c)_\theta, \bigodot^s d_0x^{m-1}\otimes\frac{\partial}{\partial y^1}_{|(0,0)}\right\rangle, \quad c\in\mathbf{R}.$$

On the other hand, given a map $h_o:\mathbf{R}\to\mathbf{R}$, we have $(h_o\cdot\Lambda)^{<o>}=h_o$. The proof of Theorem 2.2 is complete. $\square$

5. The Legendre-like operators on tuples of Lagrangians and functions on total spaces. We will use the notations as in the previous sections. We are going to prove the following:

Theorem 5.1. *Let m,n,s be positive integers. If $m\geq 3$, then any local $\mathcal{FM}_{m,n}$-natural regular operator*

$$C:\mathcal{C}^\infty_M\left(J^sY,\bigwedge^m T^*M\right)\times\mathcal{C}^\infty(Y,\mathbf{R})\to\mathcal{C}^\infty_Y\left(J^sY,S^sTM\otimes V^*Y\otimes\bigwedge^m T^*M\right)$$

is $h_o\cdot\Lambda$ for some (uniquely determined by C) map $h_o:\mathbf{R}\to\mathbf{R}$, where $h\cdot C$ is defined by

$$(h\cdot C)(\lambda,g)_{|j^s_{x_o}\sigma}=h(g(\sigma(x_o)))\cdot C(\lambda,g)_{|j^s_{x_o}\sigma}$$

for any $h:\mathbf{R}\to\mathbf{R}$ and any C in question and any $\lambda,g,j^s_{x_o}\sigma$ as above and where Λ is the Legendre operator.

So, if m,n,s are positive integers with $m\geq 3$, then the space of all C (as in Theorem 5.1) is the free 1-dimensional $\mathcal{C}^\infty(\mathbf{R})$-module and the operator Λ form the basis in this module.

Schema of the proof of Theorem 5.1. Similarly as in Lemma 4.1, C is determined by the collection of values

$$\left\langle C(\lambda,g)_\rho,\bigodot^s d_0\omega\otimes v\right\rangle\in\bigwedge^m T^*_0\mathbf{R}^m$$

for all $\lambda\in\mathcal{C}^\infty_{\mathbf{R}^m}(J^s(\mathbf{R}^{m,n}),\bigwedge^m T^*\mathbf{R}^m)$ and all $v\in T_0\mathbf{R}^n=V_{(0,0)}\mathbf{R}^{m,n}$ and all $d_0\omega\in T^*_0\mathbf{R}^m$ and all $\rho=j^s_0(\sigma)\in J^s_0(\mathbf{R}^m,\mathbf{R}^n)_0=J^s_{(0,0)}(\mathbf{R}^{m,n})$ and all $g:\mathbf{R}^{m,n}\to\mathbf{R}$.

Because of the regularity of C, we can assume that $d_{(0,0)}g(v)\neq 0$. Then using the invariance of C with respect to a $(0,0)$-preserving $\mathcal{FM}_{m,n}$-maps, we may additionally assume $g=y^1+c$ and $v=\frac{\partial}{\partial y^1}_{|(0,0)}$, where c is an arbitrary real number.

Then using the invariance of C with respect to $(x, y-\sigma(x))$, we may additionally assume $\rho = \theta := j^s_0(0)$ and $g = y^1 + \sigma^1(x) + c$. (We must replace the old additional assumption $g = y^1 + c$ by the new one $g = y^1 + \sigma^1(x) + c$.)

Further, because of the regularity of C, we can assume that $\frac{\partial}{\partial x^m}\sigma^1(0) \neq 0$. Then using the invariance of C with respect to some $(\varphi(x), y)$, we may additionally assume $g = y^1 + x^m + c$.

Next, we can write $\lambda = L((x^i),(y^j_\alpha))dx^\mu + f(x^1,\dots,x^m)dx^\mu$, where L and f are arbitrary real valued maps with $L((x^i),(0)) = 0$. Then quite similarly as in Lemma 4.5, we can write $\lambda = L((x^i),(y^j_\alpha))dx^\mu + b dx^\mu$, where L is an arbitrary real valued map with $L((x^i),(0)) = 0$ and $b \in \mathbf{R}$.

Next, quite similarly as in Lemma 4.6, we may additionally assume that $d_0\omega = d_0 x^{m-1}$.

Next, using the main result of [4], we may additionally assume that L is an arbitrary polynomial in $((x^i),(y^j_\alpha))$ of degree $\leq q$, where q is an arbitrary positive integer. Then quite similarly as in the proof of Lemma 4.7, we see that C is determined by the collection of values

$$\left\langle C(bdx^\mu, x^m + y^1 + c)_\theta, \bigodot^s d_0 x^{m-1} \otimes \frac{\partial}{\partial y^1}_{|(0,0)} \right\rangle$$

and

$$\left\langle C(ax^\beta y^1_\alpha dx^\mu + bdx^\mu, x^m + c)_\theta, \bigodot^s d_0 x^{m-1} \otimes \frac{\partial}{\partial y^1}_{|(0,0)} \right\rangle$$

for all $\alpha, \beta \in \mathbf{N}^m$ with $|\beta| \leq q$ and $|\alpha| \leq s$ and all $a, b, c \in \mathbf{R}$.

Then similarly as in Lemma 4.12, C is determined by the collection of values

$$\left\langle C(bdx^\mu, x^m + y^1 + c)_\theta, \bigodot^s d_0 x^{m-1} \otimes \frac{\partial}{\partial y^1}_{|(0,0)} \right\rangle$$

and

$$\left\langle C(y^1_{(0,\dots,0,s,0)} dx^\mu, x^m + c)_\theta, \bigodot^s d_0 x^{m-1} \otimes \frac{\partial}{\partial y^1}_{|(0,0)} \right\rangle \in \bigwedge^m T^*_0 \mathbf{R}^m$$

for all $b, c \in \mathbf{R}$.

Next, using the invariance of C with respect to

$$(x^1, x^2, \dots, x^{m-2}, \frac{1}{\tau} x^{m-1}, x^m, y^1, \dots, y^n),$$

we get

$$\tau^{s-1}\left\langle C(b\tau dx^\mu, x^m + y^1 + c)_\theta, \bigodot^s d_0 x^{m-1} \otimes \frac{\partial}{\partial y^1}_{|(0,0)} \right\rangle$$
$$= \left\langle C(bdx^\mu, x^m + y^1 + c)_\theta, \bigodot^s d_0 x^{m-1} \otimes \frac{\partial}{\partial y^1}_{|(0,0)} \right\rangle.$$

Then in the case of $s\geq 2$ we get

$$\left\langle C(bdx^{\mu},x^{m}+y^{1}+c)_{\theta},\bigodot^{s}d_{0}x^{m-1}\otimes\frac{\partial}{\partial y^{1}}_{|(0,0)}\right\rangle=0$$

and in the case of $s=1$ we get

$$\left\langle C(bdx^{\mu},x^{m}+y^{1}+c)_{\theta},\bigodot^{s}d_{0}x^{m-1}\otimes\frac{\partial}{\partial y^{1}}_{|(0,0)}\right\rangle$$
$$=\left\langle C(0dx^{\mu},x^{m}+y^{1}+c)_{\theta},\bigodot^{s}d_{0}x^{m-1}\otimes\frac{\partial}{\partial y^{1}}_{|(0,0)}\right\rangle.$$

Moreover, by the invariance of C with respect to $(\frac{1}{\tau}x^{1},x^{2},\dots,x^{m},y^{1},\dots,y^{n})$, we get

$$\left\langle C(0dx^{\mu},x^{m}+y^{1}+c)_{\theta},\bigodot^{s}d_{0}x^{m-1}\otimes\frac{\partial}{\partial y^{1}}_{|(0,0)}\right\rangle$$
$$=\tau\left\langle C(0dx^{\mu},x^{m}+y^{1}+c)_{\theta},\bigodot^{s}d_{0}x^{m-1}\otimes\frac{\partial}{\partial y^{1}}_{|(0,0)}\right\rangle.$$

Then (in both cases), we have

$$\left\langle C(bdx^{\mu},x^{m}+y^{1}+c)_{\theta},\bigodot^{s}d_{0}x^{m-1}\otimes\frac{\partial}{\partial y^{1}}_{|(0,0)}\right\rangle=0\,.$$

Consequently, C is determined by the map $C^{<o>}:\mathbf{R}\to\mathbf{R}$ defined by

$$C^{<o>}(c)dx^{\mu}_{|0}:=\left\langle C(y^{1}_{(0,\dots,0,s,0)}dx^{\mu},x^{m}+c)_{\theta},\bigodot^{s}d_{0}x^{m-1}\otimes\frac{\partial}{\partial y^{1}}_{|(0,0)}\right\rangle,$$

where $c\in\mathbf{R}$.

Conversely, given a map $h_{o}:\mathbf{R}\to\mathbf{R}$, we have $(h_{o}\cdot\Lambda)^{<o>}=h_{o}$. The proof of Theorem 5.1 is complete. □

References

[1] Kolář, I., *A geometrical version of the higher order Hamiltonian formalism in fibered manifolds*, J. Geom. Phys. **1** (1984), 127–137.

[2] Kolář, I., Michor, P. W., Slovák, J., *Natural Operations in Differential Geometry*, Springer-Verlag, Berlin–Heidelberg, 1993.

[3] Kurek, J., Mikulski, W. M., *The Euler-like operators on tuples of Lagrangians and functions on bases*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **73** (2024), 75–86.

[4] Mikulski, W. M., *On regular local operators on smooth maps*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **62**(2) (2015), 69–72.

[5] Mikulski, W. M., *On naturality of the Legendre operator*, Demonstr. Math **41**(4) (2008), 969–973.

Miroslav Doupovec
Institute of Mathematics
Brno University of Technology
Technická 2, Brno
Czech Republic
e-mail: doupovec@fme.vutbr.cz

Jan Kurek
Institute of Mathematics
Maria Curie-Sklodowska University
pl. Marii Curie-Skłodowskiej 1, Lublin
Poland
e-mail: jan.kurek@mail.umcs.pl

Włodzimierz M. Mikulski
Institute of Mathematics
Jagiellonian University
Łojasiewicza 6, Cracow
Poland
e-mail: wlodzimierz.mikulski@im.uj.edu.pl

Received September 2, 2024.