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The natural transformations between r-th order prolongation of tangent and cotangent bundles over Riemannian manifolds

Abstract. If (M, g) is a Riemannian manifold then there is the well-known base preserving vector bundle isomorphism $TM \to T^*M$ given by $v \mapsto g(v, -)$ between the tangent TM and the cotangent T^*M bundles of M. In the present note first we generalize this isomorphism to the one $J^rTM \to J^rT^*M$ between the r-th order prolongation J^rTM of tangent TM and the r-th order prolongation J^rT^*M of cotangent T^*M bundles of M. Further we describe all base preserving vector bundle maps $D_M(g) \colon J^rTM \to J^rT^*M$ depending on a Riemannian metric g in terms of natural (in g) tensor fields on M.

1. Introduction. All manifolds are smooth, Hausdorff, finite dimensional and without boundaries. Maps are assumed to be smooth, i.e. of class C^∞. Let $\mathcal{M}f_m$ denote category of m-dimensional manifolds and their embeddings.

From the general theory it is well known that the tangent TM and the cotangent T^*M bundles of M are not canonically isomorphic. However, if g is a Riemannian metric on a manifold M, there is the base preserving vector bundle isomorphism $i_g \colon TM \to T^*M$ given by $i_g(v) = g(v, -), v \in T_xM, x \in M$.

In the second section of the present note we give necessary definitions.

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In the third section first we generalize the isomorphism $i_g\colon TM \to T^*M$ depending on g to a base preserving vector bundle isomorphism $J^r i_g\colon J^rTM \to J^rT^*M$ canonically depending on g between the r-th order prolongation J^rTM of tangent TM and the r-th order prolongation J^rT^*M of cotangent T^*M bundles of M. Next we construct another more advanced base preserving vector bundle isomorphism $i_g^{<r>}\colon J^rTM \to J^rT^*M$ canonically depending on g.

In the fourth section we consider the problem of describing all $\mathcal{M}f_m$-natural operators $D\colon Riem \rightsquigarrow Hom(J^rT, J^rT^*)$ transforming Riemannian metrics g on m-dimensional manifolds M into base preserving vector bundle maps $D_M(g)\colon J^rTM \to J^rT^*M$. Our studies lead to the reduction of this problem to the one of describing all $\mathcal{M}f_m$-natural operators $t\colon Riem \rightsquigarrow T^* \otimes S^lT \otimes T^* \otimes S^kT^*$ (for $l, k = 1, \dots, r$) sending Riemannian metrics g on M into tensor fields $t_M(g)$ of types $T^* \otimes S^lT \otimes T^* \otimes S^kT^*$.

2. Definitions. Now we give some necessary definitions.

Definition 1. The r-th order prolongation of tangent bundle is a functor $J^rT\colon \mathcal{M}f_m \to \mathcal{VB}$ sending any m-manifold M into J^rTM and any embedding $\varphi\colon M_1 \to M_2$ of two manifolds into $J^rT\varphi\colon J^rTM_1 \to J^rTM_2$ given by $J^rT\varphi(j^r_xX) = j^r_{\varphi(x)}\varphi_*X$, where $X \in \mathcal{X}(M_1)$ and $\varphi_*X = T\varphi \circ X \circ \varphi^{-1}$ is the image of a vector field X by φ.

Definition 2. The r-th order prolongation of cotangent bundle is a functor $J^rT^*\colon \mathcal{M}f_m \to \mathcal{VB}$ sending any m-manifold M into J^rT^*M and any embedding $\varphi\colon M_1 \to M_2$ of two manifolds into

$$J^rT^*\varphi\colon J^rT^*M_1 \to J^rT^*M_2$$

given by $J^rT^*\varphi := J^r(T\varphi^{-1})^*$.

Definition 3. The dual bundle of the r-th order prolongation of tangent bundle is a functor $(J^rT)^*\colon \mathcal{M}f_m \to \mathcal{VB}$ sending any m-manifold M into $(J^rT)^*M := (J^rTM)^*$ and any embedding $\varphi\colon M_1 \to M_2$ of two manifolds into

$$(J^rT)^*\varphi\colon (J^rT)^*M_1 \to (J^rT)^*M_2$$

given by $(J^rT)^*\varphi := (J^rT\varphi^{-1})^*$.

Definition 4. The dual bundle of the r-th order prolongation of cotangent bundle is a functor $(J^rT^*)^*\colon \mathcal{M}f_m \to \mathcal{VB}$ sending any m-manifold M into $(J^rT^*)^*M := (J^rT^*M)^*$ and any embedding $\varphi\colon M_1 \to M_2$ of two manifolds into

$$(J^rT^*)^*\varphi\colon (J^rT^*)^*M_1 \to (J^rT^*)^*M_2$$

given by $(J^rT^*)^*\varphi := (J^rT^*\varphi^{-1})^*$.

The general concept of natural operators can be found in [4]. In particular, we have the following definitions.

Definition 5. An $\mathcal{M}f_m$-natural operator $D\colon Riem \rightsquigarrow Hom(J^rT, J^rT^*)$ transforming Riemannian metrics g on m-dimensional manifolds M into base preserving vector bundle maps $D_M(g)\colon J^rTM \to J^rT^*M$ is a system $D = \{D_M\}_{M\in obj(\mathcal{M}f_m)}$ of regular operators

$$D_M\colon Riem(M) \to Hom_M(J^rTM, J^rT^*M)$$

satisfying the $\mathcal{M}f_m$-invariance condition, where $Hom_M(J^rTM, J^rT^*M)$ is the set of all vector bundle maps $J^rTM \to J^rT^*M$ covering the identity map id_M of M.

The $\mathcal{M}f_m$-invariance condition of D is following: for any $g_1 \in Riem(M_1)$ and $g_2 \in Riem(M_2)$ if g_1 and g_2 are φ-related by an embedding $\varphi\colon M_1 \to M_2$ of m-manifolds (i.e. φ is (g_1, g_2)-isomorphism) then $D_{M_1}(g_1)$ and $D_{M_2}(g_2)$ are also φ-related (i.e. $D_{M_2}(g_2) \circ J^rT\varphi = J^rT^*\varphi \circ D_{M_1}(g_1)$).

Equivalently, the above $\mathcal{M}f_m$-invariance means that for any $g_1 \in Riem(M_1)$ and $g_2 \in Riem(M_2)$ if the diagram

(1)
$$\begin{array}{ccc} T^*M_1 \otimes T^*M_1 & \xrightarrow{T^*\varphi \otimes T^*\varphi} & T^*M_2 \otimes T^*M_2 \\ {\scriptstyle g_1}\uparrow & & \uparrow{\scriptstyle g_2} \\ M_1 & \xrightarrow[\varphi]{} & M_2 \end{array}$$

commutes for an embedding $\varphi\colon M_1 \to M_2$ (i.e. $(T^*\varphi \otimes T^*\varphi) \circ g_1 = g_2 \circ \varphi$) then the diagram

$$\begin{array}{ccc} J^rT^*M_1 & \xrightarrow{J^rT^*\varphi} & J^rT^*M_2 \\ {\scriptstyle D_{M_1}(g_1)}\uparrow & & \uparrow{\scriptstyle D_{M_2}(g_2)} \\ J^rTM_1 & \xrightarrow[J^rT\varphi]{} & J^rTM_2 \end{array}$$

commutes also.

We say that operator D_M is regular if it transforms smoothly parameterized families of Riemannian metrics into smoothly parameterized ones of vector bundle maps.

Similarly, we can define the following concepts:

- an $\mathcal{M}f_m$-natural operator $D\colon Riem \rightsquigarrow Hom(J^rT, J^rT)$,
- an $\mathcal{M}f_m$-natural operator $D\colon Riem \rightsquigarrow Hom(J^rT, (J^rT)^*)$,
- an $\mathcal{M}f_m$-natural operator $D\colon Riem \rightsquigarrow Hom(J^rT, (J^rT^*)^*)$,

- an $\mathcal{M}f_m$-natural operator $D\colon Riem \rightsquigarrow Hom(J^rT^*, J^rT)$,
- an $\mathcal{M}f_m$-natural operator $D\colon Riem \rightsquigarrow Hom(J^rT^*, J^rT^*)$,
- an $\mathcal{M}f_m$-natural operator $D\colon Riem \rightsquigarrow Hom(J^rT^*, (J^rT)^*)$,
- an $\mathcal{M}f_m$-natural operator $D\colon Riem \rightsquigarrow Hom(J^rT^*, (J^rT^*)^*)$,
- an $\mathcal{M}f_m$-natural operator $D\colon Riem \rightsquigarrow Hom((J^rT)^*, J^rT)$,
- an $\mathcal{M}f_m$-natural operator $D\colon Riem \rightsquigarrow Hom((J^rT)^*, J^rT^*)$,
- an $\mathcal{M}f_m$-natural operator $D\colon Riem \rightsquigarrow Hom((J^rT)^*, (J^rT)^*)$,
- an $\mathcal{M}f_m$-natural operator $D\colon Riem \rightsquigarrow Hom((J^rT)^*, (J^rT^*)^*)$,
- an $\mathcal{M}f_m$-natural operator $D\colon Riem \rightsquigarrow Hom((J^rT^*)^*, J^rT)$,
- an $\mathcal{M}f_m$-natural operator $D\colon Riem \rightsquigarrow Hom((J^rT^*)^*, J^rT^*)$,
- an $\mathcal{M}f_m$-natural operator $D\colon Riem \rightsquigarrow Hom((J^rT^*)^*, (J^rT)^*)$,
- an $\mathcal{M}f_m$-natural operator $D\colon Riem \rightsquigarrow Hom((J^rT^*)^*, (J^rT^*)^*)$.

Now we have the following definition.

Definition 6. An $\mathcal{M}f_m$-natural operator $A\colon Riem \rightsquigarrow (T\otimes S^lT^*, T^*\otimes S^kT^*)$ transforming Riemannian metrics g on m-dimensional manifolds M into base preserving vector bundle maps $A_M(g)\colon TM\otimes S^lT^*M \to T^*M\otimes S^kT^*M$ is a system $A = \{A_M\}_{M\in obj(\mathcal{M}f_m)}$ of regular operators $A_M\colon Riem(M) \to C^\infty(TM \otimes S^lT^*M, T^*M \otimes S^kT^*M)$ satisfying the $\mathcal{M}f_m$-invariance condition, where $C^\infty(TM\otimes S^lT^*M, T^*M\otimes S^kT^*M)$ is the set of all vector bundle maps $TM\otimes S^lT^*M \to T^*M\otimes S^kT^*M$ covering the identity map id_M of M.

The $\mathcal{M}f_m$-invariance condition of A is following : for any $g_1 \in Riem(M_1)$ and $g_2 \in Riem(M_2)$ if g_1 and g_2 are φ-related by an embedding $\varphi\colon M_1 \to M_2$ of m-manifolds (i.e. $(T^*\varphi\otimes T^*\varphi)\circ g_1 = g_2\circ\varphi$) then $A_{M_1}(g_1)$ and $A_{M_2}(g_2)$ are also φ-related (i.e. $A_{M_2}(g_2)\circ(T\varphi\otimes S^lT^*\varphi) = (T^*\varphi\otimes S^kT^*\varphi)\circ A_{M_1}(g_1))$.

Equivalently, the above $\mathcal{M}f_m$-invariance means that for any $g_1 \in Riem(M_1)$ and $g_2 \in Riem(M_2)$ if the diagram (1) commutes for an embedding $\varphi\colon M_1 \to M_2$ then the diagram

$$\begin{array}{ccc}
T^*M_1 \otimes S^kT^*M_1 & \xrightarrow{\;T^*\varphi \otimes S^kT^*\varphi\;} & T^*M_2 \otimes S^kT^*M_2 \\
{\scriptstyle A_{M_1}(g_1)}\big\uparrow & & \big\uparrow{\scriptstyle A_{M_2}(g_2)} \\
TM_1 \otimes S^lT^*M_1 & \xrightarrow[\;T\varphi \otimes S^lT^*\varphi\;]{} & TM_2 \otimes S^lT^*M_2
\end{array}$$

commutes also.

The regularity means almost the same as in Definition 5.

Similarly, we can define the following concepts:

- an $\mathcal{M}f_m$-natural operator $A\colon Riem \rightsquigarrow (T \otimes S^lT, T \otimes S^kT)$,
- an $\mathcal{M}f_m$-natural operator $A\colon Riem \rightsquigarrow (T^* \otimes S^lT, T \otimes S^kT)$,

- an $\mathcal{M}f_m$-natural operator $A\colon Riem \rightsquigarrow (T \otimes S^l T^*, T \otimes S^k T)$,
- an $\mathcal{M}f_m$-natural operator $A\colon Riem \rightsquigarrow (T \otimes S^l T, T^* \otimes S^k T)$,
- an $\mathcal{M}f_m$-natural operator $A\colon Riem \rightsquigarrow (T \otimes S^l T, T \otimes S^k T^*)$,
- an $\mathcal{M}f_m$-natural operator $A\colon Riem \rightsquigarrow (T^* \otimes S^l T^*, T \otimes S^k T)$,
- an $\mathcal{M}f_m$-natural operator $A\colon Riem \rightsquigarrow (T^* \otimes S^l T, T^* \otimes S^k T)$,
- an $\mathcal{M}f_m$-natural operator $A\colon Riem \rightsquigarrow (T^* \otimes S^l T, T \otimes S^k T^*)$,
- an $\mathcal{M}f_m$-natural operator $A\colon Riem \rightsquigarrow (T \otimes S^l T^*, T^* \otimes S^k T)$,
- an $\mathcal{M}f_m$-natural operator $A\colon Riem \rightsquigarrow (T \otimes S^l T^*, T \otimes S^k T^*)$,
- an $\mathcal{M}f_m$-natural operator $A\colon Riem \rightsquigarrow (T \otimes S^l T, T^* \otimes S^k T^*)$,
- an $\mathcal{M}f_m$-natural operator $A\colon Riem \rightsquigarrow (T^* \otimes S^l T^*, T^* \otimes S^k T)$,
- an $\mathcal{M}f_m$-natural operator $A\colon Riem \rightsquigarrow (T^* \otimes S^l T^*, T \otimes S^k T^*)$,
- an $\mathcal{M}f_m$-natural operator $A\colon Riem \rightsquigarrow (T^* \otimes S^l T, T^* \otimes S^k T^*)$,
- an $\mathcal{M}f_m$-natural operator $A\colon Riem \rightsquigarrow (T^* \otimes S^l T^*, T^* \otimes S^k T^*)$.

Next we have an important general definition of natural tensor.

Definition 7. An $\mathcal{M}f_m$-natural operator (natural tensor) $t\colon Riem \rightsquigarrow \bigotimes^p T \otimes \bigotimes^q T^*$ transforming Riemannian metrics g on m-dimensional manifolds M into tensor fields of type (p, q) on M is a system $t = \{t_M\}_{M \in obj(\mathcal{M}f_m)}$ of regular operators $t_M\colon Riem(M) \to \mathcal{T}^{(p,q)}(M)$ satisfying the $\mathcal{M}f_m$-invariance condition, where $\mathcal{T}^{(p,q)}(M)$ is the set of tensor fields of type (p, q) on M.

The $\mathcal{M}f_m$-invariance condition of t is following : for any $g_1 \in Riem(M_1)$ and $g_2 \in Riem(M_2)$ if g_1 and g_2 are φ-related by an embedding $\varphi\colon M_1 \to M_2$ of m-manifolds (i.e. $(T^*\varphi \otimes T^*\varphi) \circ g_1 = g_2 \circ \varphi$) then $t_{M_1}(g_1)$ and $t_{M_2}(g_2)$ are also φ-related (i.e. $t_{M_2}(g_2) \circ \varphi = (\bigotimes^p T\varphi \otimes \bigotimes^q T^*\varphi) \circ t_{M_1}(g_1))$.

Equivalently, the above $\mathcal{M}f_m$-invariance means that for any $g_1 \in Riem(M_1)$ and $g_2 \in Riem(M_2)$ if the diagram (1) commutes for an embedding $\varphi\colon M_1 \to M_2$, then the diagram

$$\begin{array}{ccc}
\bigotimes^p TM_1 \otimes \bigotimes^q T^*M_1 & \xrightarrow{\bigotimes^p T\varphi \otimes \bigotimes^q T^*\varphi} & \bigotimes^p TM_2 \otimes \bigotimes^q T^*M_2 \\
\uparrow {\scriptstyle t_{M_1}(g_1)} & & \uparrow {\scriptstyle t_{M_2}(g_2)} \\
M_1 & \xrightarrow[\varphi]{} & M_2
\end{array}$$

commutes also.

We say that operator t_M is regular if it transforms smoothly parametrized families of Riemannian metrics into smoothly parametrized ones of tensor fields.

Now we have a definition of a special kind of natural tensor.

Definition 8. An $\mathcal{M}f_m$-natural operator (natural tensor) $t\colon Riem \rightsquigarrow T^* \otimes S^lT \otimes T^* \otimes S^kT^*$ transforming Riemannian metrics g on m-dimensional manifolds M into tensor fields of type $T^*\otimes S^lT\otimes T^*\otimes S^kT^*$ on M is a system $t = \{t_M\}_{M\in obj(\mathcal{M}f_m)}$ of regular operators $t_M\colon Riem(M) \to C^\infty(T^*M \otimes S^lTM \otimes T^*M \otimes S^kT^*M)$ satisfying the $\mathcal{M}f_m$-invariance condition, where $C^\infty(T^*M \otimes S^lTM \otimes T^*M \otimes S^kT^*M)$ is the set of all tensor fields of type $T^* \otimes S^lT \otimes T^* \otimes S^kT^*$ on M.

The $\mathcal{M}f_m$-invariance condition of t is following: for any $g_1 \in Riem(M_1)$ and $g_2 \in Riem(M_2)$ if g_1 and g_2 are φ-related by an embedding $\varphi\colon M_1 \to M_2$ of m-manifolds (i.e. $(T^*\varphi\otimes T^*\varphi)\circ g_1 = g_2\circ\varphi$), then $t_{M_1}(g_1)$ and $t_{M_2}(g_2)$ are also φ-related (i.e. $t_{M_2}(g_2)\circ\varphi = (T^*\varphi\otimes S^lT\varphi\otimes T^*\varphi\otimes S^kT^*\varphi)\circ t_{M_1}(g_1)$).

Equivalently, the above $\mathcal{M}f_m$-invariance means that for any $g_1 \in Riem(M_1)$ and $g_2 \in Riem(M_2)$ if the diagram (1) commutes for an embedding $\varphi\colon M_1 \to M_2$, then the diagram

$$\begin{array}{ccc}
T^*M_1\otimes S^lTM_1\otimes T^*M_1\otimes S^kT^*M_1 & \xrightarrow{\ \Phi\ } & T^*M_2\otimes S^lTM_2\otimes T^*M_2\otimes S^kT^*M_2 \\
\uparrow {\scriptstyle t_{M_1}(g_1)} & & \uparrow {\scriptstyle t_{M_2}(g_2)} \\
M_1 & \xrightarrow[\ \varphi\ ]{} & M_2
\end{array}$$

commutes also, where $\Phi = T^*\varphi \otimes S^lT\varphi \otimes T^*\varphi \otimes S^kT^*\varphi$.

The regularity means almost the same as in Definition 7.

Similarly, we can define the following concepts:

- an $\mathcal{M}f_m$-natural operator (natural tensor) $t\colon Riem \rightsquigarrow T \otimes S^lT \otimes T \otimes S^kT$,
- an $\mathcal{M}f_m$-natural operator (natural tensor) $t\colon Riem \rightsquigarrow T^* \otimes S^lT \otimes T \otimes S^kT$,
- an $\mathcal{M}f_m$-natural operator (natural tensor) $t\colon Riem \rightsquigarrow T \otimes S^lT^* \otimes T \otimes S^kT$,
- an $\mathcal{M}f_m$-natural operator (natural tensor) $t\colon Riem \rightsquigarrow T \otimes S^lT \otimes T^* \otimes S^kT$,
- an $\mathcal{M}f_m$-natural operator (natural tensor) $t\colon Riem \rightsquigarrow T \otimes S^lT \otimes T \otimes S^kT^*$,
- an $\mathcal{M}f_m$-natural operator (natural tensor) $t\colon Riem \rightsquigarrow T^* \otimes S^lT^* \otimes T \otimes S^kT$,
- an $\mathcal{M}f_m$-natural operator (natural tensor) $t\colon Riem \rightsquigarrow T^* \otimes S^lT \otimes T^* \otimes S^kT$,

- an $\mathcal{M}f_m$-natural operator (natural tensor) $t\colon Riem \rightsquigarrow T^* \otimes S^lT \otimes T \otimes S^kT^*$,
- an $\mathcal{M}f_m$-natural operator (natural tensor) $t\colon Riem \rightsquigarrow T \otimes S^lT^* \otimes T^* \otimes S^kT$,
- an $\mathcal{M}f_m$-natural operator (natural tensor) $t\colon Riem \rightsquigarrow T \otimes S^lT^* \otimes T \otimes S^kT^*$,
- an $\mathcal{M}f_m$-natural operator (natural tensor) $t\colon Riem \rightsquigarrow T \otimes S^lT \otimes T^* \otimes S^kT^*$,
- an $\mathcal{M}f_m$-natural operator (natural tensor) $t\colon Riem \rightsquigarrow T^* \otimes S^lT^* \otimes T^* \otimes S^kT$,
- an $\mathcal{M}f_m$-natural operator (natural tensor) $t\colon Riem \rightsquigarrow T^* \otimes S^lT^* \otimes T \otimes S^kT^*$,
- an $\mathcal{M}f_m$-natural operator (natural tensor) $t\colon Riem \rightsquigarrow T \otimes S^lT^* \otimes T^* \otimes S^kT^*$,
- an $\mathcal{M}f_m$-natural operator (natural tensor) $t\colon Riem \rightsquigarrow T^* \otimes S^lT^* \otimes T^* \otimes S^kT^*$.

In the third section we present also explicit examples of $\mathcal{M}f_m$-natural operators $D\colon Riem \rightsquigarrow Hom(J^rT, J^rT^*)$.

A full description of all polynomial natural tensors $t\colon Riem \rightsquigarrow \bigotimes^p T \otimes \bigotimes^q T^*$ transforming Riemannian metrics on m-manifolds into tensor fields of types (p,q) can be found in [1]. This description is following. Each covariant derivative of the curvature $\mathcal{R}(g) \in \mathcal{T}^{(0,4)}(M)$ of a Riemannian metric g is a natural tensor and the metric g is also a natural tensor. Further all the natural tensors $t\colon Riem \rightsquigarrow \bigotimes^p T \otimes \bigotimes^q T^*$ can be obtained by a procedure:

(a) every tensor multiplication of two natural tensors give a new natural tensor,
(b) every contraction on one covariant and one contravariant entry of a natural tensor give a new natural tensor,
(c) we can tensorize any natural tensor with a metric independent natural tensor,
(d) we can permute any number of entries in the tensor product,
(e) we can repeat these steps,
(f) we can take linear combinations.

Furthermore, if we take respective type natural tensors and apply respective symmetrization, then we can produce many natural tensors $t\colon Riem \rightsquigarrow T^* \otimes S^lT \otimes T^* \otimes S^kT^*$.

3. Constructions.

Example 1. Let (M,g) be a Riemannian manifold. Then we have a base preserving vector bundle isomorphism $i_g\colon TM \to T^*M$ given by

$$i_g(v) = g(v,-),\ v \in T_xM,\ x \in M.$$

Next we can obtain a base preserving vector bundle isomorphism $J^r i_g\colon J^rTM \to J^rT^*M$ defined by a formula

$$J^r i_g(j^r_x X) = j^r_x(i_g \circ X),$$

where $X \in \mathcal{X}(M)$. Similarly we receive also a base preserving vector bundle isomorphism

$$(J^r i_g^{-1})^*\colon (J^rTM)^* \to (J^rT^*M)^*.$$

Because of the canonical character of the above constructions we get the following propositions.

Proposition 1. *The family* $A^{(r)}\colon Riem \rightsquigarrow Hom(J^rT, J^rT^*)$ *of operators*

$$A^{(r)}_M\colon Riem(M) \to Hom_M(J^rTM, J^rT^*M), \quad A^{(r)}_M(g) = J^r i_g$$

for all $M \in obj(\mathcal{M}f_m)$ *is an* $\mathcal{M}f_m$*-natural operator.*

Proposition 2. *The family* $A^{[r]}\colon Riem \rightsquigarrow Hom((J^rT)^*, (J^rT^*)^*)$ *of operators*

$$A^{[r]}_M\colon Riem(M) \to Hom_M((J^rTM)^*, (J^rT^*M)^*), \quad A^{[r]}_M(g) = (J^r i_g^{-1})^*$$

for all $M \in obj(\mathcal{M}f_m)$ *is an* $\mathcal{M}f_m$*-natural operator.*

Now we are going to present another more advanced example of an $\mathcal{M}f_m$-natural operator $D\colon Riem \rightsquigarrow Hom(J^rT, J^rT^*)$.

Recall that if g is a Riemannian tensor field on a manifold M and $x \in M$, then there is g-normal coordinate system $\varphi\colon (M,x) \to (\mathbb{R}^m, 0)$ with centre x. If $\psi\colon (M,x) \to (\mathbb{R}^m, 0)$ is another g-normal coordinate system with centre x, then there is $A \in O(m)$ such that $\psi = A \circ \varphi$ near x. Let $I\colon J^r_0 T\mathbb{R}^m \to \bigoplus_{k=1}^r T_0\mathbb{R}^m \otimes S^k T^*_0\mathbb{R}^m = \bigoplus_{k=0}^r \mathbb{R}^m \otimes S^k\mathbb{R}^{m*}$ (see [3]) and $I_1\colon J^r_0 T^*\mathbb{R}^m \to \bigoplus_{k=1}^r T^*_0\mathbb{R}^m \otimes S^k T^*_0\mathbb{R}^m = \bigoplus_{k=0}^r \mathbb{R}^{m*} \otimes S^k\mathbb{R}^{m*}$ (see [7]) be the standard $O(m)$-invariant vector space isomorphisms.

We have the following important proposition.

Proposition 3. *Let* g *be a Riemannian tensor field on a manifold* M*. Then there are (canonical in* g*) vector bundle isomorphisms*

$$I_g\colon J^rTM \to \bigoplus_{k=0}^r TM \otimes S^kT^*M,$$

$$J_g\colon J^rT^*M \to \bigoplus_{k=0}^r T^*M \otimes S^kT^*M,$$

$$(I_g^{-1})^*\colon (J^rTM)^* \to \left(\bigoplus_{k=0}^r TM \otimes S^kT^*M\right)^* \cong \bigoplus_{k=0}^r T^*M \otimes S^kTM,$$

$$(J_g^{-1})^*\colon (J^rT^*M)^* \to \left(\bigoplus_{k=0}^r T^*M \otimes S^kT^*M\right)^* \cong \bigoplus_{k=0}^r TM \otimes S^kTM.$$

Proof. Let $v = j^r_x X \in J^r_x TM$, where $X \in \mathcal{X}(M)$, $x \in M$. Let $\varphi\colon (M,x) \to (\mathbb{R}^m, 0)$ be a g-normal coordinate system with centre x. We define

$$I_g(v) := I^\varphi_g(v) = \left(\bigoplus_{k=0}^{r} T\varphi^{-1} \otimes S^k T^* \varphi^{-1}\right) \circ I \circ J^r T\varphi(v).$$

If $\psi\colon (M,x) \to (\mathbb{R}^m, 0)$ is another g-normal coordinate system with centre x, then $\psi = A \circ \varphi$ (near x) for some $A \in O(m)$. The $O(m)$-invariance of I means that

$$I \circ J^r TA = \left(\bigoplus_{k=0}^{r} T_0 A \otimes S^k T^*_0 A\right) \circ I. \tag{2}$$

Hence we deduce that

$$\begin{aligned}
I^\psi_g(v) &= \left(\bigoplus_{k=0}^{r} T\psi^{-1} \otimes S^k T^* \psi^{-1}\right) \circ I \circ J^r T\psi(v)\\
&= \bigoplus_{k=0}^{r} (T(A\circ\varphi)^{-1} \otimes S^k T^*(A\circ\varphi)^{-1}) \circ I \circ J^r T(A\circ\varphi)(v)\\
&= \bigoplus_{k=0}^{r} ((T\varphi^{-1} \circ TA^{-1}) \otimes S^k T^*(\varphi^{-1}\circ A^{-1})) \circ I \circ (J^r TA \circ J^r T\varphi)(v)\\
&= \bigoplus_{k=0}^{r} ((T\varphi^{-1} \circ TA^{-1}) \otimes S^k T^*(\varphi^{-1}\circ A^{-1})) \circ (I \circ J^r TA) \circ J^r T\varphi(v) =: L.
\end{aligned}$$

Now using (2), we receive

$$\begin{aligned}
L &= \bigoplus_{k=0}^{r} ((T\varphi^{-1}\circ TA^{-1}) \otimes S^k T^*(\varphi^{-1}\circ A^{-1})) \circ \left(\bigoplus_{k=0}^{r} TA \otimes S^k T^* A\right) \circ I \circ J^r T\varphi(v)\\
&= \bigoplus_{k=0}^{r} \Big[((T\varphi^{-1}\circ TA^{-1})\circ TA) \otimes (S^k T^*(\varphi^{-1}\circ A^{-1})\circ S^k T^* A)\Big] \circ I \circ J^r T\varphi(v)\\
&= \bigoplus_{k=0}^{r} ((T\varphi^{-1} \circ TA^{-1} \circ TA) \otimes S^k T^*(\varphi^{-1} \circ A^{-1} \circ A)) \circ I \circ J^r T\varphi(v)\\
&= \bigoplus_{k=0}^{r} (T\varphi^{-1} \otimes S^k T^* \varphi^{-1}) \circ I \circ J^r T\varphi(v) = I^\varphi_g(v).
\end{aligned}$$

Therefore, the definition of $I_g(v)$ is independent of the choice of φ. So, isomorphism $I_g\colon J^r TM \to \bigoplus_{k=0}^r TM \otimes S^k T^* M$ is well defined.

Similarly, we put

$$J_g(v) := J^\varphi_g(v) = \left(\bigoplus_{k=0}^{r} T^*\varphi^{-1} \otimes S^k T^* \varphi^{-1}\right) \circ I_1 \circ J^r T^* \varphi(v).$$

Using $O(m)$-invariance of I_1 (i.e. $I_1 \circ J^rT^*A = (\bigoplus_{k=0}^r T_0^*A \otimes S^kT_0^*A) \circ I_1$) analogously as before, we show that $I_g^\psi(v) = I_g^\varphi(v)$. This proves that the definition of $J_g(v)$ is independent of the choice of g-normal coordinate system φ with centre x and the isomorphism $J_g \colon J^rT^*M \to \bigoplus_{k=0}^r T^*M \otimes S^kT^*M$ is well defined.

Finally we obtain (canonical in g) vector bundle isomorphisms

$$(I_g^{-1})^* \colon (J^rTM)^* \to \left(\bigoplus_{k=0}^r TM \otimes S^kT^*M\right)^* \cong \bigoplus_{k=0}^r T^*M \otimes S^kTM$$

$$(J_g^{-1})^* \colon (J^rT^*M)^* \to \left(\bigoplus_{k=0}^r T^*M \otimes S^kT^*M\right)^* \cong \bigoplus_{k=0}^r TM \otimes S^kTM. \quad \square$$

Remark 1. W. Mikulski (in [7]) has recently constructed a (canonical in ∇) vector bundle isomorphism $I_\nabla \colon J^rTM \to \bigoplus_{k=0}^r T^*M \otimes S^kT^*M$ for a classical linear connection ∇ on a manifold M.

Now we have further important identifications.

Example 2. Let (M,g) be a Riemannian manifold and $i_g \colon TM \to T^*M$ be a base preserving vector bundle isomorphism recalled in Example 1. Using the base preserving vector bundle isomorphisms I_g and J_g from Proposition 3, we receive the following vector bundle isomorphisms

$$i_g^{<r>} := J_g^{-1} \circ \left(\bigoplus_{k=0}^r i_g \otimes S^kT^*id_M\right) \circ I_g \colon J^rTM \to J^rT^*M,$$

$$i_g^{[r]} := J_g^* \circ \left(\bigoplus_{k=0}^r i_g^{-1} \otimes S^kTid_M\right) \circ (I_g^{-1})^* \colon (J^rTM)^* \to (J^rT^*M)^*.$$

Because of canonical character of above constructions we obtain the following propositions.

Proposition 4. *The family $B^{<r>} \colon Riem \rightsquigarrow Hom(J^rT, J^rT^*)$ of operators*

$$B_M^{<r>} \colon Riem(M) \to Hom_M(J^rTM, J^rT^*M), \quad B_M^{<r>}(g) = i_g^{<r>}$$

for all $M \in obj(\mathcal{M}f_m)$ is an $\mathcal{M}f_m$-natural operator.

Proposition 5. *The family $B^{[r]} \colon Riem \rightsquigarrow Hom((J^rT)^*, (J^rT^*)^*)$ of operators*

$$B_M^{[r]} \colon Riem(M) \to Hom_M((J^rTM)^*, (J^rT^*M)^*), \quad B_M^{[r]}(g) = i_g^{[r]}$$

for all $M \in obj(\mathcal{M}f_m)$ is an $\mathcal{M}f_m$-natural operator.

Example 3. Let (M,g) be a Riemannian manifold. In an article [5] J. Kurek and W. Mikulski constructed a base preserving vector bundle isomorphism

$$i_g^{(r)} \colon \bigoplus_{k=1}^r S^kTM \to \bigoplus_{k=1}^r S^kT^*M$$

given by

$$i_g^{(r)}(v_1 \odot \cdots \odot v_k) = i_g(v_1) \odot \cdots \odot i_g(v_k).$$

Now using this isomorphism, we get a base preserving vector bundle isomorphism

$$I_g^{(r)} \colon TM \otimes \bigoplus_{k=0}^{r} S^kT^*M \cong \bigoplus_{k=0}^{r} TM \otimes S^kT^*M$$
$$\to T^*M \otimes \bigoplus_{k=0}^{r} S^kTM \cong \bigoplus_{k=0}^{r} T^*M \otimes S^kTM$$

defined by a formula

$$I_g^{(r)} = i_g \otimes (i_g^{(r)})^{-1}.$$

Similarly, we construct another base preserving vector bundle isomorphisms

$$\tilde{I}_g^{(r)} \colon \bigoplus_{k=0}^{r} TM \otimes S^kT^*M \to \bigoplus_{k=0}^{r} TM \otimes S^kTM, \quad \tilde{I}_g^{(r)} = id_{TM} \otimes \left(i_g^{(r)}\right)^{-1},$$
$$\tilde{\tilde{I}}_g^{(r)} \colon \bigoplus_{k=0}^{r} T^*M \otimes S^kT^*M \to \bigoplus_{k=0}^{r} TM \otimes S^kTM, \quad \tilde{\tilde{I}}_g^{(r)} = i_g^{-1} \otimes \left(i_g^{(r)}\right)^{-1},$$
$$\hat{I}_g^{(r)} \colon \bigoplus_{k=0}^{r} T^*M \otimes S^kT^*M \to \bigoplus_{k=0}^{r} T^*M \otimes S^kTM, \quad \hat{I}_g^{(r)} = id_{T^*M} \otimes \left(i_g^{(r)}\right)^{-1}.$$

Thus we receive a base preserving vector bundle isomorphism

$$I_g^{<r>} \colon J^rTM \to (J^rTM)^*$$

given by

$$I_g^{<r>} = I_g^* \circ I_g^{(r)} \circ I_g.$$

Similarly, we construct another base preserving vector bundle isomorphisms

$$\tilde{I}_g^{<r>} \colon J^rTM \to (J^rT^*M)^*, \quad \tilde{I}_g^{<r>} = J_g^* \circ \tilde{I}_g^{(r)} \circ I_g,$$
$$I_g^{[r]} \colon J^rT^*M \to (J^rTM)^*, \quad I_g^{[r]} = I_g^* \circ \hat{I}_g^{(r)} \circ J_g,$$
$$\tilde{I}_g^{[r]} \colon J^rT^*M \to (J^rT^*M)^*, \quad \tilde{I}_g^{[r]} = J_g^* \circ \tilde{\tilde{I}}_g^{(r)} \circ J_g.$$

Using the base preserving vector bundle isomorphism $J^ri_g \colon J^rTM \to J^rT^*M$ constructed in Example 1, we obtain also the following vector bundle isomorphisms

$$J_g^{<r>} = (J^ri_g^{-1})^* \circ I_g^{<r>} \colon J^rTM \to (J^rT^*M)^*,$$
$$\tilde{J}_g^{<r>} = (J^ri_g)^* \circ \tilde{I}_g^{<r>} \colon J^rTM \to (J^rTM)^*,$$
$$J_g^{[r]} = (J^ri_g^{-1})^* \circ I_g^{[r]} \colon J^rT^*M \to (J^rT^*M)^*,$$
$$\tilde{J}_g^{[r]} = (J^ri_g)^* \circ \tilde{I}_g^{[r]} \colon J^rT^*M \to (J^rTM)^*.$$

Because of canonical character of the above constructions we get the following propositions.

Proposition 6. *The family* $C^{<r>}\colon Riem \rightsquigarrow Hom(J^rT, (J^rT)^*)$ *of operators*

$$C^{<r>}_M\colon Riem(M) \to Hom_M(J^rTM, (J^rTM)^*), \quad C^{<r>}_M(g) = I^{<r>}_g$$

for all $M \in obj(\mathcal{M}f_m)$ *is an* $\mathcal{M}f_m$*-natural operator.*

Proposition 7. *The family* $\tilde{C}^{<r>}\colon Riem \rightsquigarrow Hom(J^rT, (J^rT^*)^*)$ *of operators*

$$\tilde{C}^{<r>}_M\colon Riem(M) \to Hom_M(J^rTM, (J^rT^*M)^*), \quad \tilde{C}^{<r>}_M(g) = \tilde{I}^{<r>}_g$$

for all $M \in obj(\mathcal{M}f_m)$ *is an* $\mathcal{M}f_m$*-natural operator.*

Proposition 8. *The family* $C^{[r]}\colon Riem \rightsquigarrow Hom(J^rT^*, (J^rT)^*)$ *of operators*

$$C^{[r]}_M\colon Riem(M) \to Hom_M(J^rT^*M, (J^rTM)^*), \quad C^{[r]}_M(g) = I^{[r]}_g$$

for all $M \in obj(\mathcal{M}f_m)$ *is an* $\mathcal{M}f_m$*-natural operator.*

Proposition 9. *The family* $\tilde{C}^{[r]}\colon Riem \rightsquigarrow Hom(J^rT^*, (J^rT^*)^*)$ *of operators*

$$\tilde{C}^{[r]}_M\colon Riem(M) \to Hom_M(J^rT^*M, (J^rT^*M)^*), \quad \tilde{C}^{[r]}_M(g) = \tilde{I}^{[r]}_g$$

for all $M \in obj(\mathcal{M}f_m)$ *is an* $\mathcal{M}f_m$*-natural operator.*

Proposition 10. *The family* $D^{<r>}\colon Riem \rightsquigarrow Hom(J^rT, (J^rT^*)^*)$ *of operators*

$$D^{<r>}_M\colon Riem(M) \to Hom_M(J^rTM, (J^rT^*M)^*), \quad D^{<r>}_M(g) = J^{<r>}_g$$

for all $M \in obj(\mathcal{M}f_m)$ *is an* $\mathcal{M}f_m$*-natural operator.*

Proposition 11. *The family* $\tilde{D}^{<r>}\colon Riem \rightsquigarrow Hom(J^rT, (J^rT)^*)$ *of operators*

$$\tilde{D}^{<r>}_M\colon Riem(M) \to Hom_M(J^rTM, (J^rTM)^*), \quad \tilde{D}^{<r>}_M(g) = \tilde{J}^{<r>}_g$$

for all $M \in obj(\mathcal{M}f_m)$ *is an* $\mathcal{M}f_m$*-natural operator.*

Proposition 12. *The family* $D^{[r]}\colon Riem \rightsquigarrow Hom(J^rT^*, (J^rT^*)^*)$ *of operators*

$$D^{[r]}_M\colon Riem(M) \to Hom_M(J^rT^*M, (J^rT^*M)^*), \quad D^{[r]}_M(g) = J^{[r]}_g$$

for all $M \in obj(\mathcal{M}f_m)$ *is an* $\mathcal{M}f_m$*-natural operator.*

Proposition 13. *The family* $\tilde{D}^{[r]}\colon Riem \rightsquigarrow Hom(J^rT^*, (J^rT)^*)$ *of operators*

$$\tilde{D}^{[r]}_M\colon Riem(M) \to Hom_M(J^rT^*M, (J^rTM)^*), \quad \tilde{D}^{[r]}_M(g) = \tilde{J}^{[r]}_g$$

for all $M \in obj(\mathcal{M}f_m)$ *is an* $\mathcal{M}f_m$*-natural operator.*

4. The main results. Let $g \in Riem(M)$ be a Riemannian metric on an m-manifold M. By Proposition 3 and Examples 1, 2, 3 we have identifications

$$J^rTM = J^rT^*M = (J^rTM)^* = (J^rT^*M)^* = \bigoplus_{k=0}^{r} TM \otimes S^kT^*M$$
$$= \bigoplus_{k=0}^{r} T^*M \otimes S^kT^*M = \bigoplus_{k=0}^{r} T^*M \otimes S^kTM = \bigoplus_{k=0}^{r} TM \otimes S^kTM$$

modulo the base preserving vector bundle isomorphisms canonically depending on g.

Consequently, the problem of finding all $\mathcal{M}f_m$-natural operators $D\colon Riem \rightsquigarrow Hom(J^rT, J^rT^*)$ is reduced to the one of finding all systems $(A^{l,k})$ of $\mathcal{M}f_m$-natural operators $A^{l,k}\colon Riem \rightsquigarrow (T \otimes S^lT^*, T^* \otimes S^kT^*)$ transforming Riemannian metrics g on m-manifolds M into base preserving vector bundle maps $A^{l,k}_M(g)\colon TM \otimes S^lT^*M \to T^*M \otimes S^kT^*M$, where $l, k = 1, \dots, r$ or (equivalently) our problem is reduced to the one of finding all natural tensors $t^{l,k}\colon Riem \rightsquigarrow T^* \otimes S^lT \otimes T^* \otimes S^kT^*$ transforming Riemannian metrics g on m-manifolds M into tensor fields of types $T^* \otimes S^lT \otimes T^* \otimes S^kT^*$ on M for $l, k = 1, \dots, r$.

Thus we have proved the following theorem.

Theorem 1. *The $\mathcal{M}f_m$-natural operators $D\colon Riem \rightsquigarrow Hom(J^rT, J^rT^*)$ transforming Riemannian metrics g on m-manifolds M into base preserving vector bundle maps $D_M(g)\colon J^rTM \to J^rT^*M$ are in the bijection with systems $(t^{l,k})$ of $\mathcal{M}f_m$-natural operators (natural tensors) $t^{l,k}\colon Riem \rightsquigarrow T^* \otimes S^lT \otimes T^* \otimes S^kT^*$ transforming Riemannian metrics g on m-manifolds M into tensor fields of types $T^* \otimes S^lT \otimes T^* \otimes S^kT^*$ on M for $l, k = 1, \dots, r$.*

Because of the isomorphism $J^rTM \cong J^rT^*M$ depending on g, we have the following theorem.

Theorem 2. *The $\mathcal{M}f_m$-natural operators $D\colon Riem \rightsquigarrow Hom(J^rT, J^rT)$ transforming Riemannian metrics g on m-manifolds M into base preserving vector bundle maps $D_M(g)\colon J^rTM \to J^rTM$ are in the bijection with systems $(A^{l,k})$ of $\mathcal{M}f_m$-natural operators $A^{l,k}\colon Riem \rightsquigarrow (T \otimes S^lT^*, T \otimes S^kT^*)$ transforming Riemannian metrics g on m-manifolds M into base preserving vector bundle maps $A^{l,k}_M(g)\colon TM \otimes S^lT^*M \to TM \otimes S^kT^*M$ for $l, k = 1, \dots, r$ or (equivalently) in the bijection with systems $(t^{l,k})$ of $\mathcal{M}f_m$-natural operators (natural tensors) $t^{l,k}\colon Riem \rightsquigarrow T^* \otimes S^lT \otimes T \otimes S^kT^*$ transforming Riemannian metrics g on m-manifolds M into tensor fields of types $T^* \otimes S^lT \otimes T \otimes S^kT^*$ on M for $l, k = 1, \dots, r$.*

By the same reason, we have also the following corollary.

Corollary 1. *The $\mathcal{M}f_m$-natural operators $D\colon Riem \rightsquigarrow Hom(J^rT^*, J^rT)$ transforming Riemannian metrics g on m-manifolds M into base preserving* *vector bundle maps* $D_M(g)\colon J^rT^*M \to J^rTM$ *are in the bijection with systems* $(A^{l,k})$ *of* $\mathcal{M}f_m$*-natural operators* $A^{l,k}\colon Riem \rightsquigarrow (T^*\otimes S^lT^*, T\otimes S^kT^*)$ *transforming Riemannian metrics* g *on* m*-manifolds* M *into base preserving vector bundle maps* $A^{l,k}_M(g)\colon T^*M\otimes S^lT^*M \to TM\otimes S^kT^*M$ *for* $l,k=1,\dots,r$ *or (equivalently) in the bijection with systems* $(t^{l,k})$ *of* $\mathcal{M}f_m$*-natural operators (natural tensors)* $t^{l,k}\colon Riem \rightsquigarrow T\otimes S^lT\otimes T\otimes S^kT^*$ *transforming Riemannian metrics* g *on* m*-manifolds* M *into tensor fields of types* $T\otimes S^lT\otimes T\otimes S^kT^*$ *on* M *for* $l,k=1,\dots,r$.

By the same reason, we have another corollary.

Corollary 2. *The* $\mathcal{M}f_m$*-natural operators* $D\colon Riem \rightsquigarrow Hom(J^rT^*, J^rT^*)$ *transforming Riemannian metrics* g *on* m*-manifolds* M *into base preserving vector bundle maps* $D_M(g)\colon J^rT^*M \to J^rT^*M$ *are in the bijection with systems* $(A^{l,k})$ *of* $\mathcal{M}f_m$*-natural operators* $A^{l,k}\colon Riem \rightsquigarrow (T^*\otimes S^lT^*, T^*\otimes S^kT^*)$ *transforming Riemannian metrics* g *on* m*-manifolds* M *into base preserving vector bundle maps* $A^{l,k}_M(g)\colon T^*M\otimes S^lT^*M \to T^*M\otimes S^kT^*M$ *for* $l,k=1,\dots,r$ *or (equivalently) in the bijection with systems* $(t^{l,k})$ *of* $\mathcal{M}f_m$*-natural operators (natural tensors)* $t^{l,k}\colon Riem \rightsquigarrow T\otimes S^lT\otimes T^*\otimes S^kT^*$ *transforming Riemannian metrics* g *on* m*-manifolds* M *into tensor fields of types* $T\otimes S^lT\otimes T^*\otimes S^kT^*$ *on* M *for* $l,k=1,\dots,r$.

Because of the isomorphisms $J^rTM \cong J^rT^*M \cong (J^rTM)^*$ depending on g, we have the following theorem.

Theorem 3. *The* $\mathcal{M}f_m$*-natural operators* $D\colon Riem \rightsquigarrow Hom(J^rT, (J^rT)^*)$ *transforming Riemannian metrics* g *on* m*-manifolds* M *into base preserving vector bundle maps* $D_M(g)\colon J^rTM \to (J^rTM)^*$ *are in the bijection with systems* $(A^{l,k})$ *of* $\mathcal{M}f_m$*-natural operators* $A^{l,k}\colon Riem \rightsquigarrow (T\otimes S^lT^*, T^*\otimes S^kT)$ *transforming Riemannian metrics* g *on* m*-manifolds* M *into base preserving vector bundle maps* $A^{l,k}_M(g)\colon TM\otimes S^lT^*M \to T^*M\otimes S^kTM$ *for* $l,k=1,\dots,r$ *or (equivalently) in the bijection with systems* $(t^{l,k})$ *of* $\mathcal{M}f_m$*-natural operators (natural tensors)* $t^{l,k}\colon Riem \rightsquigarrow T^*\otimes S^lT\otimes T^*\otimes S^kT$ *transforming Riemannian metrics* g *on* m*-manifolds* M *into tensor fields of types* $T^*\otimes S^lT\otimes T^*\otimes S^kT$ *on* M *for* $l,k=1,\dots,r$.

By the same reason, we have the following theorem.

Theorem 4. *The* $\mathcal{M}f_m$*-natural operators* $D\colon Riem \rightsquigarrow Hom(J^rT^*, (J^rT)^*)$ *transforming Riemannian metrics* g *on* m*-manifolds* M *into base preserving vector bundle maps* $D_M(g)\colon J^rT^*M \to (J^rTM)^*$ *are in the bijection with systems* $(A^{l,k})$ *of* $\mathcal{M}f_m$*-natural operators* $A^{l,k}\colon Riem \rightsquigarrow (T^*\otimes S^lT^*, T^*\otimes S^kT)$ *transforming Riemannian metrics* g *on* m*-manifolds* M *into base preserving vector bundle maps* $A^{l,k}_M(g)\colon T^*M\otimes S^lT^*M \to T^*M\otimes S^kTM$ *for*

$l, k = 1, \dots, r$ or (equivalently) in the bijection with systems $(t^{l,k})$ of $\mathcal{M}f_m$-natural operators (natural tensors) $t^{l,k}\colon Riem \rightsquigarrow T \otimes S^l T \otimes T^* \otimes S^k T$ transforming Riemannian metrics g on m-manifolds M into tensor fields of types $T \otimes S^l T \otimes T^* \otimes S^k T$ on M for $l, k = 1, \dots, r$.

By the same reason, we have also the following corollary.

Corollary 3. *The $\mathcal{M}f_m$-natural operators $D\colon Riem \rightsquigarrow Hom((J^rT)^*, J^rT)$ transforming Riemannian metrics g on m-manifolds M into base preserving vector bundle maps $D_M(g)\colon (J^rTM)^* \to J^rTM$ are in the bijection with systems $(A^{l,k})$ of $\mathcal{M}f_m$-natural operators $A^{l,k}\colon Riem \rightsquigarrow (T^* \otimes S^l T, T \otimes S^k T^*)$ transforming Riemannian metrics g on m-manifolds M into base preserving vector bundle maps $A^{l,k}_M(g)\colon T^*M \otimes S^l TM \to TM \otimes S^k T^*M$ for $l, k = 1, \dots, r$ or (equivalently) in the bijection with systems $(t^{l,k})$ of $\mathcal{M}f_m$-natural operators (natural tensors) $t^{l,k}\colon Riem \rightsquigarrow T \otimes S^l T^* \otimes T \otimes S^k T^*$ transforming Riemannian metrics g on m-manifolds M into tensor fields of types $T \otimes S^l T^* \otimes T \otimes S^k T^*$ on M for $l, k = 1, \dots, r$.*

We have also the next similar corollary.

Corollary 4. *The $\mathcal{M}f_m$-natural operators $D\colon Riem \rightsquigarrow Hom((J^rT)^*, J^rT^*)$ transforming Riemannian metrics g on m-manifolds M into base preserving vector bundle maps $D_M(g)\colon (J^rTM)^* \to J^rT^*M$ are in the bijection with systems $(A^{l,k})$ of $\mathcal{M}f_m$-natural operators $A^{l,k}\colon Riem \rightsquigarrow (T^* \otimes S^l T, T^* \otimes S^k T^*)$ transforming Riemannian metrics g on m-manifolds M into base preserving vector bundle maps $A^{l,k}_M(g)\colon T^*M \otimes S^l TM \to T^*M \otimes S^k T^*M$ for $l, k = 1, \dots, r$ or (equivalently) in the bijection with systems $(t^{l,k})$ of $\mathcal{M}f_m$-natural operators (natural tensors) $t^{l,k}\colon Riem \rightsquigarrow T \otimes S^l T^* \otimes T^* \otimes S^k T^*$ transforming Riemannian metrics g on m-manifolds M into tensor fields of types $T \otimes S^l T^* \otimes T^* \otimes S^k T^*$ on M for $l, k = 1, \dots, r$.*

We have also another corollary.

Corollary 5. *The $\mathcal{M}f_m$-natural operators $D\colon Riem \rightsquigarrow Hom((J^rT)^*, (J^rT)^*)$ transforming Riemannian metrics g on m-manifolds M into base preserving vector bundle maps $D_M(g)\colon (J^rTM)^* \to (J^rTM)^*$ are in the bijection with systems $(A^{l,k})$ of $\mathcal{M}f_m$-natural operators $A^{l,k}\colon Riem \rightsquigarrow (T^* \otimes S^l T, T^* \otimes S^k T)$ transforming Riemannian metrics g on m-manifolds M into base preserving vector bundle maps $A^{l,k}_M(g)\colon T^*M \otimes S^l TM \to T^*M \otimes S^k TM$ for $l, k = 1, \dots, r$ or (equivalently) in the bijection with systems $(t^{l,k})$ of $\mathcal{M}f_m$-natural operators (natural tensors) $t^{l,k}\colon Riem \rightsquigarrow T \otimes S^l T^* \otimes T^* \otimes S^k T$ transforming Riemannian metrics g on m-manifolds M into tensor fields of types $T \otimes S^l T^* \otimes T^* \otimes S^k T$ on M for $l, k = 1, \dots, r$.*

Because of the isomorphisms $J^rTM \cong J^rT^*M \cong (J^rTM)^* \cong (J^rT^*M)^*$ depending on g, we have the following theorem.

Theorem 5. *The $\mathcal{M}f_m$-natural operators $D\colon Riem \rightsquigarrow Hom(J^rT, (J^rT^*)^*)$ transforming Riemannian metrics g on m-manifolds M into base preserving vector bundle maps $D_M(g)\colon J^rTM \to (J^rT^*M)^*$ are in the bijection with systems $(A^{l,k})$ of $\mathcal{M}f_m$-natural operators $A^{l,k}\colon Riem \rightsquigarrow (T\otimes S^lT^*, T\otimes S^kT)$ transforming Riemannian metrics g on m-manifolds M into base preserving vector bundle maps $A^{l,k}_M(g)\colon TM\otimes S^lT^*M \to TM\otimes S^kTM$ for $l,k = 1,\dots,r$ or (equivalently) in the bijection with systems $(t^{l,k})$ of $\mathcal{M}f_m$-natural operators (natural tensors) $t^{l,k}\colon Riem \rightsquigarrow T^*\otimes S^lT\otimes T\otimes S^kT$ transforming Riemannian metrics g on m-manifolds M into tensor fields of types $T^*\otimes S^lT\otimes T\otimes S^kT$ on M for $l,k = 1,\dots,r$.*

By the same reason, we have the following theorem.

Theorem 6. *The $\mathcal{M}f_m$-natural operators $D\colon Riem \rightsquigarrow Hom(J^rT^*, (J^rT^*)^*)$ transforming Riemannian metrics g on m-manifolds M into base preserving vector bundle maps $D_M(g)\colon J^rT^*M \to (J^rT^*M)^*$ are in the bijection with systems $(A^{l,k})$ of $\mathcal{M}f_m$-natural operators $A^{l,k}\colon Riem \rightsquigarrow (T^*\otimes S^lT^*, T\otimes S^kT)$ transforming Riemannian metrics g on m-manifolds M into base preserving vector bundle maps $A^{l,k}_M(g)\colon T^*M\otimes S^lT^*M \to TM\otimes S^kTM$ for $l,k = 1,\dots,r$ or (equivalently) in the bijection with systems $(t^{l,k})$ of $\mathcal{M}f_m$-natural operators (natural tensors) $t^{l,k}\colon Riem \rightsquigarrow T\otimes S^lT\otimes T\otimes S^kT$ transforming Riemannian metrics g on m-manifolds M into tensor fields of types $T\otimes S^lT\otimes T\otimes S^kT$ on M for $l,k = 1,\dots,r$.*

By the same reason, we have also the following theorem.

Theorem 7. *The $\mathcal{M}f_m$-natural operators $D\colon Riem \rightsquigarrow Hom((J^rT)^*, (J^rT^*)^*)$ transforming Riemannian metrics g on m-manifolds M into base preserving vector bundle maps $D_M(g)\colon (J^rTM)^* \to (J^rT^*M)^*$ are in the bijection with systems $(A^{l,k})$ of $\mathcal{M}f_m$-natural operators $A^{l,k}\colon Riem \rightsquigarrow (T^*\otimes S^lT, T\otimes S^kT)$ transforming Riemannian metrics g on m-manifolds M into base preserving vector bundle maps $A^{l,k}_M(g)\colon T^*M\otimes S^lTM \to TM\otimes S^kTM$ for $l,k = 1,\dots,r$ or (equivalently) in the bijection with systems $(t^{l,k})$ of $\mathcal{M}f_m$-natural operators (natural tensors) $t^{l,k}\colon Riem \rightsquigarrow T\otimes S^lT^*\otimes T\otimes S^kT$ transforming Riemannian metrics g on m-manifolds M into tensor fields of types $T\otimes S^lT^*\otimes T\otimes S^kT$ on M for $l,k = 1,\dots,r$.*

We have also the following corollary.

Corollary 6. *The $\mathcal{M}f_m$-natural operators $D\colon Riem \rightsquigarrow Hom((J^rT^*)^*, J^rT)$ transforming Riemannian metrics g on m-manifolds M into base preserving vector bundle maps $D_M(g)\colon (J^rT^*M)^* \to J^rTM$ are in the bijection with systems $(A^{l,k})$ of $\mathcal{M}f_m$-natural operators $A^{l,k}\colon Riem \rightsquigarrow (T\otimes S^lT, T\otimes S^kT^*)$ transforming Riemannian metrics g on m-manifolds M into base preserving vector bundle maps $A^{l,k}_M(g)\colon TM\otimes S^lTM \to TM\otimes S^kT^*M$ for $l,k = 1,\dots,r$*

or (equivalently) in the bijection with systems $(t^{l,k})$ *of* $\mathcal{M}f_m$*-natural operators (natural tensors)* $t^{l,k}\colon Riem \rightsquigarrow T^*\otimes S^lT^*\otimes T\otimes S^kT^*$ *transforming Riemannian metrics* g *on* m*-manifolds* M *into tensor fields of types* $T^*\otimes S^lT^*\otimes T\otimes S^kT^*$ *on* M *for* $l,k=1,\dots,r$.

We have also the next corollary.

Corollary 7. *The* $\mathcal{M}f_m$*-natural operators* $D\colon Riem \rightsquigarrow Hom((J^rT^*)^*, J^rT^*)$ *transforming Riemannian metrics* g *on* m*-manifolds* M *into base preserving vector bundle maps* $D_M(g)\colon (J^rT^*M)^*\to J^rT^*M$ *are in the bijection with systems* $(A^{l,k})$ *of* $\mathcal{M}f_m$*-natural operators* $A^{l,k}\colon Riem \rightsquigarrow (T\otimes S^lT, T^*\otimes S^kT^*)$ *transforming Riemannian metrics* g *on* m*-manifolds* M *into base preserving vector bundle maps* $A^{l,k}_M(g)\colon TM\otimes S^lTM\to T^*M\otimes S^kT^*M$ *for* $l,k=1,\dots,r$ *or (equivalently) in the bijection with systems* $(t^{l,k})$ *of* $\mathcal{M}f_m$*-natural operators (natural tensors)* $t^{l,k}\colon Riem \rightsquigarrow T^*\otimes S^lT^*\otimes T^*\otimes S^kT^*$ *transforming Riemannian metrics* g *on* m*-manifolds* M *into tensor fields of types* $T^*\otimes S^lT^*\otimes T^*\otimes S^kT^*$ *on* M *for* $l,k=1,\dots,r$.

We have also the similar corollary.

Corollary 8. *The* $\mathcal{M}f_m$*-natural operators* $D\colon Riem \rightsquigarrow Hom((J^rT^*)^*, (J^rT)^*)$ *transforming Riemannian metrics* g *on* m*-manifolds* M *into base preserving vector bundle maps* $D_M(g)\colon (J^rT^*M)^*\to (J^rTM)^*$ *are in the bijection with systems* $(A^{l,k})$ *of* $\mathcal{M}f_m$*-natural operators* $A^{l,k}\colon Riem \rightsquigarrow (T\otimes S^lT, T^*\otimes S^kT)$ *transforming Riemannian metrics* g *on* m*-manifolds* M *into base preserving vector bundle maps* $A^{l,k}_M(g)\colon TM\otimes S^lTM\to T^*M\otimes S^kTM$ *for* $l,k=1,\dots,r$ *or (equivalently) in the bijection with systems* $(t^{l,k})$ *of* $\mathcal{M}f_m$*-natural operators (natural tensors)* $t^{l,k}\colon Riem \rightsquigarrow T^*\otimes S^lT^*\otimes T^*\otimes S^kT$ *transforming Riemannian metrics* g *on* m*-manifolds* M *into tensor fields of types* $T^*\otimes S^lT^*\otimes T^*\otimes S^kT$ *on* M *for* $l,k=1,\dots,r$.

Finally, we have the last corollary.

Corollary 9. *The* $\mathcal{M}f_m$*-natural operators* $D\colon Riem \rightsquigarrow Hom((J^rT^*)^*, (J^rT^*)^*)$ *transforming Riemannian metrics* g *on* m*-manifolds* M *into base preserving vector bundle maps* $D_M(g)\colon (J^rT^*M)^*\to (J^rT^*M)^*$ *are in the bijection with systems* $(A^{l,k})$ *of* $\mathcal{M}f_m$*-natural operators* $A^{l,k}\colon Riem \rightsquigarrow (T\otimes S^lT, T\otimes S^kT)$ *transforming Riemannian metrics* g *on* m*-manifolds* M *into base preserving vector bundle maps* $A^{l,k}_M(g)\colon TM\otimes S^lTM\to TM\otimes S^kTM$ *for* $l,k=1,\dots,r$ *or (equivalently) in the bijection with systems* $(t^{l,k})$ *of* $\mathcal{M}f_m$*-natural operators (natural tensors)* $t^{l,k}\colon Riem \rightsquigarrow T^*\otimes S^lT^*\otimes T\otimes S^kT$ *transforming Riemannian metrics* g *on* m*-manifolds* M *into tensor fields of types* $T^*\otimes S^lT^*\otimes T\otimes S^kT$ *on* M *for* $l,k=1,\dots,r$.

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