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On intermediate *q*-Lauricella functions in the spirit of Karlsson, Chandel Singh and Gupta

Abstract. The purpose of this article is to define some intermediate q-Lauricella functions, to find convergence regions in two different forms, and to prove corresponding reduction formulas by using a known lemma from our first book. These convergence regions are given in form of q-additions and q-real numbers. The third q-real number plays a special role in the computations. Generating functions are proved by using the q-binomial theorem. Finally, special cases of q-Lauricella functions as well as confluent forms in the spirit of Chandel Singh and Gupta are given.

Contents

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1. Introduction

This paper is part of a series of papers on multiple q-hypergeometric functions, where hypergeometric formulas are simply q-deformed by the author's logarithmic notation. The same goes for the convergence regions, where he author's q-real numbers are used. That is why these q-real numbers are introduced in Section 2. Many attempts have been made to generalize the Appell, Lauricella and Lauricella triple functions by Exton [9], [10], by Srivastava [13], [14] (to four variables) and by Qurechi et al. [12] (to four variables), etc. The most successful generalization was by Karlsson [11], who introduced "symmetric" intermediate Lauricella functions. It seems that "more general forms" of multiple hypergeometric functions, which lack symmetry, do not give concise, short formulas.

Therefore, we strictly follow Karlsson's paper [11] and show that it leads to nice formulas and convergence regions. In the process, one of his formulas is corrected. In order to do this, we need to remind the reader of the q-real numbers from [8]. First, however, we present some notation from [2].

Definition 1. Let $\delta > 0$ be an arbitrary small number. We will always use the following branch of the logarithm: $-\pi + \delta < \mathrm{Im}(\log q) \leq \pi + \delta$. This defines a simply connected space in the complex plane.

The power function is defined by

$$q^a \equiv e^{a \log(q)}.$$

The q-shifted factorial is defined by

$$\langle a; q\rangle_n \equiv \prod_{m=0}^{n-1} (1 - q^{a+m}).$$

Definition 2. A q-analogue of [1, p. 198]:

$$\begin{aligned}&{}^{(k)}\Phi_{\mathrm{AC}}^{(n)}(a, b, b_{k+1}, \ldots, b_n; c_1, \ldots, c_n \mid q; x_1, \ldots, x_n)\\ &\equiv \sum_{\vec{m}} \frac{\langle a; q\rangle_m \langle b; q\rangle_{m_1+\ldots+m_k} \prod_{j=k+1}^{n} \langle b_j; q\rangle_{m_j}}{\prod_{j=1}^{n} \langle c_j; q\rangle_{m_j} \langle \vec{1}; q\rangle_{\vec{m}}} \vec{x}^{\vec{m}}.\end{aligned}$$

A q-analogue of [1, p. 198]:

$$\begin{aligned}&{}^{(k)}\Phi_{\mathrm{AD}}^{(n)}(a, b_1, \ldots, b_n; c, c_{k+1}, \ldots, c_n \mid q; x_1, \ldots, x_n)\\ &\equiv \sum_{\vec{m}} \frac{\langle a; q\rangle_m \prod_{j=1}^{n} \langle b_j; q\rangle_{m_j}}{\langle c; q\rangle_{m_1+\ldots+m_k} \langle \vec{1}; q\rangle_{\vec{m}} \prod_{j=k+1}^{n} \langle c_j; q\rangle_{m_j}} \vec{x}^{\vec{m}}.\end{aligned}$$

A q-analogue of [1, p. 198]:

$$\begin{aligned}&{}^{(k)}\Phi_{\mathrm{BD}}^{(n)}(a, a_{k+1}, \ldots, a_n, b_1, \ldots, b_n; c \mid q; x_1, \ldots, x_n)\\ &\equiv \sum_{\vec{m}} \frac{\langle a; q\rangle_{m_1+\ldots+m_k} \prod_{j=k+1}^{n} \langle a_j; q\rangle_{m_j} \prod_{j=1}^{n} \langle b_j; q\rangle_{m_j}}{\langle c; q\rangle_m \langle \vec{1}; q\rangle_{\vec{m}}} \vec{x}^{\vec{m}}.\end{aligned}$$

A q-analogue of [11, p. 212]:

$$
\begin{aligned}
&{}^{(k)}\Phi_{\mathrm{CD}}^{(n)}(a,b,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_n\,|\,q;x_1,\ldots,x_n)\\
&\equiv \sum_{\vec{m}} \frac{\langle a;q\rangle_m \langle b;q\rangle_{m_{k+1}+\ldots+m_n} \prod_{j=1}^{k}\langle b_j;q\rangle_{m_j}}{\langle c;q\rangle_{m_1+\ldots+m_k}\langle \vec{1};q\rangle_{\vec{m}} \prod_{j=k+1}^{n}\langle c_j;q\rangle_{m_j}} \vec{x}^{\vec{m}}.
\end{aligned}
$$

The function ${}^{(k)}\Phi_{\mathrm{AC}}^{(n)}$ contains $\Phi_{\mathrm{A}}^{(n)}(a,\vec{b};\vec{c}\mid\vec{x})$ in the special case $k=0$, and $\Phi_{\mathrm{C}}^{(n)}(a,b;\vec{c}\,|\,q;\vec{x})$ in the special case $k=n$.

The function ${}^{(k)}\Phi_{\mathrm{AD}}^{(n)}$ contains $\Phi_{\mathrm{A}}^{(n)}(a,\vec{b};\vec{c}\,|\,q;\vec{x})$ in the special case $k=0$, and $\Phi_{\mathrm{D}}^{(n)}(a,\vec{b};c\,|\,q;\vec{x})$ in the special case $k=n$.

The function ${}^{(k)}\Phi_{\mathrm{BD}}^{(n)}$ contains $\Phi_{\mathrm{B}}^{(n)}(\vec{a},\vec{b};c\,|\,q;\vec{x})$ in the special case $k=0$, and $\Phi_{\mathrm{D}}^{(n)}(a,\vec{b};c\,|\,q;\vec{x})$ in the special case $k=n$.

The function ${}^{(k)}\Phi_{\mathrm{CD}}^{(n)}$ contains $\Phi_{\mathrm{C}}^{(n)}(a,b;\vec{c}\,|\,q;\vec{x})$ in the special case $k=0$, and $\Phi_{\mathrm{D}}^{(n)}(a,\vec{b};c\,|\,q;\vec{x})$ in the special case $k=n$.

2. Survey of q-real numbers

The q-real numbers give a convenient notation for q-additions in formal power series, in particular for q-exponential and q-trigonometric functions. There is a one-to-one correspondence between the convergence regions of the two q-Lauricella functions $\Phi_{\mathrm{A}}^{(n)}$ and $\Phi_{\mathrm{C}}^{(n)}$ [3, 4] and the existence of q-real numbers with n letters (or variables). There are three types of q-real numbers [8]: $\mathbb{R}_{\oplus_q}$, compare with [5], $\mathbb{R}_q$ and $\mathbb{R}_{\boxplus_q}$.

Definition 3 ([2, p. 24]). Let a and b be elements of a commutative semigroup. Then the NWA q-addition is given by

$$
(a \oplus_q b)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q a^k b^{n-k},\ n \in \mathbb{N}_0.
$$

In particular, $(a \oplus_q b)^0 \equiv 1$. Furthermore, we put

$$
(a \ominus_q b)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q a^k (-b)^{n-k},\ n \in \mathbb{N}_0.
$$

Definition 4. Let a and b be elements of a commutative semigroup. The Jackson–Hahn–Cigler q-addition (JHC) is the function

$$
(a \boxplus_q b)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q q^{\binom{k}{2}}\, b^k a^{n-k} = a^n \left(-\frac{b}{a};q\right)_n,\ n \in \mathbb{N}_0.
$$

The JHC q-subtraction is defined analogously:

$$
(a \boxminus_q b)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q q^{\binom{k}{2}}\, (-b)^k a^{n-k},\ n \in \mathbb{N}_0.
$$

We just give a brief survey of these three q-real number definitions from [8].

Definition 5 ([2]). The q-multinomial coefficient is defined by

$$\binom{n}{k_1, k_2, \dots, k_m}_q \equiv \frac{\langle 1;q\rangle_n}{\langle 1;q\rangle_{k_1}\langle 1;q\rangle_{k_2}\dots\langle 1;q\rangle_{k_m}}, \tag{1}$$

where $k_1+k_2+\dots+k_m=n$. If the number of k_i is unspecified for $m=\infty$ in (1), we denote the q-multinomial coefficients by

$$\binom{n}{\vec{k}}_q, \quad \sum_{i=1}^{\infty} k_i = n.$$

For $\vec{m}\in\mathbb{N}^n$ put

$$|\vec{m}| \equiv m_1+\dots+m_n.$$

The q-real number $\mathbb{R}_{\oplus_q}$, which appears inside the paranthesis of (2), is defined by

$$(a_1 \oplus_q a_2 \oplus_q \dots \oplus_q a_n)^k \equiv \sum_{\vec{|m|}=k}\prod_{l=1}^{n}(a_l)^{m_l}\binom{k}{\vec{m}}_q. \tag{2}$$

The q-real number $\mathbb{R}_q$, which appears inside the paranthesis of (3), is defined by

$$\begin{aligned} F(k) &\equiv (a_1 \oplus_q a_2 \oplus_q \dots \oplus_q a_n)^k \\ &\equiv \sum_{|\vec{m}|=k}\prod_{l=1}^{n}(a_l)^{m_l}\binom{k}{\vec{m}}_q, \quad \oplus_q \equiv \vee \oplus_q \vee \ominus_q \vee \boxplus_q \vee \boxminus_q . \end{aligned} \tag{3}$$

In formula (3) we have to multiply every term $(a_l)^{m_l}$ by $(-1)^{m_l}q^{\binom{m_l}{2}}$ if a minus and/or a $\boxplus_q$ is preceded by a_l in $F(k)$.

Definition 6. Assume that $\vec{m}\equiv(m_1,\dots,m_n)$, $m\equiv m_1+\dots+m_n$ and $a\in\mathbb{R}^\star$. The vector q-multinomial-coefficient $\binom{a}{\vec{m}}_q^\star$ is defined by the symmetric expression

$$\binom{a}{\vec{m}}_q^\star \equiv \frac{\langle -a;q\rangle_m(-1)^m q^{-\binom{\vec{m}}{2}+am}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\dots\langle 1;q\rangle_{m_n}}.$$

Definition 7. Let the JHC q-real numbers $\mathbb{R}_{\boxplus_q}$ with $n+1$ letters be defined as follows:

$$\mathbb{R}_{\boxplus_q} \equiv \{1 \boxminus_q q^a x_1 \boxminus_q \cdots \boxminus_q q^a x_n\}, \tag{4}$$

$\{x_k\}_1^n \in \mathbb{R}$, $a\in\mathbb{R}^\star$, $|x_k|<1$, $0<q<1$. When any x_k is negative, we replace $\boxminus_q$ by $\boxplus_q$. This means that the JHC q-real numbers in (4) are functions of $n+1$ real numbers $\{x_k\}_1^n$, a.

The following formula applies to a q-deformed hypercube of length 1 in $\mathbb{R}^n$.

Definition 8. Assuming that the right-hand side converges, and $a \in \mathbb{R}^\star$:

$$(5)\quad (1 \boxminus_q q^a x_1 \boxminus_q \cdots \boxminus_q q^a x_n)^{-a} \equiv \sum_{m_1,\ldots,m_n=0}^{\infty} \prod_{j=1}^{n} (-x_j)^{m_j} \binom{-a}{\vec{m}}_q^\star q^{\binom{\vec{m}}{2}+am}.$$

The q-real number in (4) exists only when the series (5) or (6) converges.

The following formula applies to a q-deformed hyper-rhombus of length 1 in $\mathbb{R}^n$.

Corollary 2.1. *A generalization of the q-binomial theorem:*

$$(6)\qquad (1 \boxminus_q q^a x_1 \boxminus_q \cdots \boxminus_q q^a x_n)^{-a} = \sum_{\vec{m}=\vec{0}}^{\vec{\infty}} \frac{\langle a; q\rangle_m \vec{x}^{\vec{m}}}{\langle \vec{1}; q\rangle_{\vec{m}}},\ a \in \mathbb{R}^\star.$$

3. Convergence regions

Consider the first q-real number x_q. When we write

$$x_q \equiv |x_1| \oplus_q \ldots \oplus_q |x_n| < 1,$$

[3, 4], we mean

$$(|x_1| \oplus_q \ldots \oplus_q |x_n|)^k < 1,\ k \in \mathbb{N},\ k > k_0,$$

where k_0 is the supposed maximum exponent of x_q, compare [2].

We shall give convergence regions in two different forms; one of these forms use the third q-real numbers.

Put [11]

$$K \equiv \sum_{i=1}^{k} m_i,\ N \equiv \sum_{j=k+1}^{n} m_j.$$

Theorem 3.1. *The convergence region for ${}^{(k)}\Phi_{\mathrm{AC}}^{(n)}$ is*

$$(1 \boxminus_q q^{1+K} x_{k+1} \boxminus_q \ldots \boxminus_q q^{1+K} x_n)^{-1-K} < 1,$$
$$\sqrt{|x_1|} \oplus_q \ldots \oplus_q \sqrt{|x_k|} < 1.$$

For q close to 1, this can also be described as

$$(\sqrt{|x_1|} \oplus_q \ldots \oplus_q \sqrt{|x_k|})^2 \oplus_q |x_{k+1}| \oplus_q \cdots \oplus_q |x_n| < 1.$$

Proof. Put all parameters equal to unity and consider the series

$$\sigma_1 \equiv \sum_{\vec{m}} \frac{\langle 1;q\rangle_{K+N}\langle 1;q\rangle_K}{(\prod_{j=1}^{k}\langle 1;q\rangle_{m_j})^2\langle\vec{1};q\rangle_{\vec{m}}\prod_{j=k+1}^{n}\langle 1;q\rangle_{m_j}}|\vec{x}|^{\vec{m}}$$
$$= \sum_{m_1,\ldots,m_k=0}^{\infty}\left[\frac{\langle 1;q\rangle_K}{\prod_{j=1}^{k}\langle 1;q\rangle_{m_j}}\right]^2\prod_{j=1}^{k}|x_j|^{m_j}$$
$$\times \sum_{m_{k+1},\ldots,m_n=0}^{\infty}\frac{\langle 1+K;q\rangle_N\prod_{j=k+1}^{n}|x_j|^{m_j}}{\prod_{j=k+1}^{n}\langle 1;q\rangle_{m_j}}.$$

The second series is equal to

$$(1 \boxminus_q q^{1+K}x_{k+1} \boxminus_q \cdots \boxminus_q q^{1+K}x_n)^{-1-K}$$

and the first one is

$$\Phi_{\mathrm{C}}^{(k)}(1,1,1,\ldots,1;1,\ldots,1\,|\,q;|x_1|,\ldots,|x_k|),$$

with known convergence region. □

Theorem 3.2. *The convergence region for* ${}^{(k)}\Phi_{\mathrm{AD}}^{(n)}$ *is*

$$(1 \boxminus_q q^{1+K}x_{k+1} \boxminus_q \ldots \boxminus_q q^{1+K}x_n)^{-1-K} < 1.$$

For q close to 1, this can also be described as

$$(\max(|x_1|,\ldots,|x_k|)) \oplus_q |x_{k+1}| \oplus_q \ldots \oplus_q |x_n| < 1.$$

Proof.

$$\sigma_2 \equiv \sum_{\vec{m}} \frac{\langle 1;q\rangle_{K+N}}{\langle 1;q\rangle_K\prod_{j=k+1}^{n}\langle 1;\rangle_{m_j}}|\vec{x}|^{\vec{m}}$$
$$= \sum_{m_1,\ldots,m_k=0}^{\infty}\prod_{j=1}^{k}|x_j|^{m_j}\sum_{m_{k+1},\ldots,m_n=0}^{\infty}\frac{\langle 1+K;q\rangle_N\prod_{j=k+1}^{n}|x_j|^{m_j}}{\prod_{j=k+1}^{n}\langle 1;q\rangle_{m_j}}.$$

The second series is again equal to

$$(1 \boxminus_q q^{1+K}x_{k+1} \boxminus_q \ldots \boxminus_q q^{1+K}x_n)^{-1-K}.$$ □

Theorem 3.3. *The convergence region for* ${}^{(k)}\Phi_{\mathrm{BD}}^{(n)}$ *is*

$$\max(|x_1|,\ldots,|x_n|) < 1.$$

Proof. Put

$$\sigma_3 \equiv \sum_{\vec{m}} \frac{\langle 1;q\rangle_K\prod_{j=k+1}^{n}\langle 1;q\rangle_{m_j}}{\langle 1;q\rangle_{K+N}}|\vec{x}|^{\vec{m}}.$$

Clearly,

$$\sigma_3 < \sum_{\vec{m}}|\vec{x}|^{\vec{m}},$$

which converges for

$$\max(|x_1|, \ldots, |x_n|) < 1. \qquad \square$$

Theorem 3.4. *The convergence region for* ${}^{(k)}\Phi_{\mathrm{CD}}^{(n)}$ *is*

$$\max(|x_1|, \ldots, |x_k|) < 1, \; \sqrt{|x_{k+1}|} \oplus_q \ldots \oplus_q \sqrt{|x_n|} < 1.$$

For q close to 1, this can also be described as

$$(\max(|x_1|, \ldots, |x_k|)) \oplus_q (\sqrt{|x_{k+1}|} \oplus_q \ldots \oplus_q \sqrt{|x_n|})^2 < 1.$$

Proof. Put

$$Y \equiv \max(|x_1|, \ldots, |x_k|).$$

Consider the series

$$\sigma_4 \equiv \sum_{m_{k+1},\ldots,m_n=0}^{\infty} \left[\frac{\langle 1; q\rangle_N}{\prod_{j=k+1}^{n} \langle 1; q\rangle_{m_j}} \right]^2 \prod_{j=k+1}^{n} |x_j|^{m_j} \sigma_5.$$

We have put

$$\sigma_5 = \sum_{m_1,\ldots,m_k=0}^{\infty} \frac{\langle 1+N; q\rangle_K \prod_{j=1}^{k} |x_j|^{m_j}}{\langle 1; q\rangle_K}$$
$$= \sum_{p=0}^{\infty} \frac{\langle 1+N; q\rangle_p}{\langle 1; q\rangle_p} \sum_{K=p} \prod_{j=1}^{k} |x_j|^{m_j}.$$

Again, one of the terms in the last series is Y^p. Then we have

$$\sigma_5 > \sum_{p=0}^{\infty} \frac{\langle 1+N; q\rangle_p Y^p}{\langle 1; q\rangle_p} = \frac{1}{(Y; q)_{1+N}},$$

which implies

$$\sigma_4 > \frac{1}{(Y; q)_{1+N}} \Phi_{\mathrm{C}}^{(n-k)}(1, 1; 1, \ldots, 1 \mid q; |x_{k+1}|, \ldots, |x_n|).$$

As before, the number of terms in the sum with $K = p$ is less than $(p+1)^k$, and for every $\epsilon > 0$ there is $A > 0$ such that $(p+1)^k < A(1+\epsilon)^p$, $\forall p > 0$. This implies

$$\sigma_5 < \sum_{p=0}^{\infty} \frac{\langle 1+N; q\rangle_p A(1+\epsilon)^p Y^p}{\langle 1; q\rangle_p} = A \frac{1}{((1+\epsilon)Y; q)_{1+N}}$$

and

$$\sigma_4 < A \frac{1}{((1+\epsilon)Y; q)_{1+N}} \Phi_{\mathrm{C}}^{(n-k)}(1, 1; 1, \ldots, 1 \mid q; |x_{k+1}|, \ldots, |x_n|).$$

Since ϵ is arbitrary small, because of the double inequality for σ_4, we obtain the stated convergence regions. $\square$

4. Reducible cases

There are three reducible cases, for $\Phi_{\mathrm{AD}}^{(n)}$, $\Phi_{\mathrm{BD}}^{(n)}$ and $\Phi_{\mathrm{CD}}^{(n)}$, which all use the following reducibility case from [2]. We shall use the following abbreviations:

$$b \equiv \sum_{i=1}^{k} b_i, \;\; N \equiv \sum_{j=k+1}^{n} m_j,$$

where $\{m_j\}_{j=k+1}^{n}$ are the summation indices in the corresponding multiple q-series.

Lemma 4.1 ([2, 10.143]).

$$\begin{aligned} &\Phi_{\mathrm{D}}^{(k)}(A, b_1, \ldots, b_k; C \mid q; xq^{C-A-b+b_2+\ldots+b_k}, \\ &\quad xq^{C-A-b+b_3+\ldots+b_k}, \ldots, xq^{C-A-b}) \\ &= {}_2\phi_1\left(A, b; C \mid q; xq^{C-A-b}\right). \end{aligned} \tag{7}$$

Theorem 4.2. *A reduction formula for* ${}^{(k)}\Phi_{\mathrm{AD}}^{(n)}$ *and a q-analogue of* [11, p. 219]*:*

$$\begin{aligned} &{}^{(k)}\Phi_{\mathrm{AD}}^{(n)}\left[a, b_1, \ldots, b_n; c, c_{k+1}, \ldots, c_n \;\middle|\; q; xq^{c-a-b-N+b_2+\ldots+b_k}, \right. \\ &\quad \left. xq^{c-a-b-N+b_3+\ldots+b_k}, \ldots, xq^{c-a-b-N}, x_{k+1}, \ldots, x_n \right] \\ &= \Phi_{\mathrm{A}}^{(n-k+1)}\left[\begin{matrix} a, b_1+\ldots+b_k, b_{k+1}, \ldots, b_n \\ c, c_{k+1}, \ldots, c_n \end{matrix} \;\middle|\; q; xq^{c-a-b-N}, \right. \\ &\quad \left. x_{k+1}, \ldots, x_n \right]. \end{aligned}$$

Proof.

$$\begin{aligned} \mathrm{LHS} &= \sum_{m_{k+1},\ldots,m_n} \frac{\langle a; q\rangle_{m_{k+1}+\ldots+m_n} \prod_{j=k+1}^{n} \langle b_j; q\rangle_{m_j} x_j^{m_j}}{\prod_{j=k+1}^{n} \langle c_j, 1; q\rangle_{m_j}} \\ &\Phi_{\mathrm{D}}^{(k)}\left[\begin{matrix} a+m_{k+1}+\ldots+m_n, b_1, \ldots, b_k \\ c \end{matrix} \;\middle|\; q; xq^{c-a-b-N+b_2+\ldots+b_k}, \right. \\ &\quad \left. xq^{c-a-b-N+b_3+\ldots+b_k}, \ldots, xq^{c-a-b-N} \right] \\ &\overset{\text{by (7)}}{=} \sum_{m_{k+1},\ldots,m_n} \frac{\langle a; q\rangle_{m_{k+1}+\ldots+m_n} \prod_{j=k+1}^{n} \langle b_j; q\rangle_{m_j} x_j^{m_j}}{\prod_{j=k+1}^{n} \langle c_j, 1; q\rangle_{m_j}} \\ &\quad \times {}_2\phi_1\left(a+m_{k+1}+\ldots+m_n, b; c \mid q; xq^{c-a-b-N}\right) = \mathrm{RHS}. \qquad \square \end{aligned}$$

Theorem 4.3. *A reduction formula for* ${}^{(k)}\Phi_{\mathrm{BD}}^{(n)}$ *and a q-analogue of the corrected version of* [11, p. 220 (5.2)]*:*

$$
\begin{aligned}
&{}^{(k)}\Phi_{\mathrm{BD}}^{(n)}\left[a, a_{k+1}, \ldots, a_n, b_1, \ldots, b_n; c \;\middle|\; q; xq^{c-a-b+N+b_2+\ldots+b_k}, \right.\\
&\qquad \left. xq^{c-a-b+N+b_3+\ldots+b_k}, \ldots, xq^{c-a-b+N}, x_{k+1}, \ldots, x_n \right]\\
&= {}^{(1)}\Phi_{\mathrm{BD}}^{(n-k+1)}\left[\begin{matrix} a, a_{k+1}, \ldots, a_n, b_1 + \ldots + b_k, b_{k+1}, \ldots, b_n \\ c \end{matrix} \;\middle|\; q; \right.\\
&\qquad \left. xq^{c-a-b+N}, x_{k+1}, \ldots, x_n \right].
\end{aligned}
$$

Proof.

$$
\begin{aligned}
&\mathrm{LHS} = \sum_{m_{k+1},\ldots,m_n} \frac{\prod_{j=k+1}^{n} \langle a_j; q\rangle_{m_j} \langle b_j; q\rangle_{m_j} x_j^{m_j}}{\langle c; q\rangle_{m_{k+1}+\ldots+m_n} \prod_{j=1}^{n} \langle 1; q\rangle_{m_j}}\\
&\Phi_{\mathrm{D}}^{(k)}\left[\begin{matrix} a, b_1, \ldots, b_k \\ c + m_{k+1} + \ldots + m_n \end{matrix} \;\middle|\; q; xq^{c-a-b+N+b_2+\cdots+b_k}, \right.\\
&\qquad \left. xq^{c-a-b+N+b_3+\ldots+b_k}, \ldots, xq^{c-a-b+N} \right]\\
&\overset{\text{by (7)}}{=} \sum_{m_{k+1},\ldots,m_n} \frac{\prod_{j=k+1}^{n} \langle a_j, b_j; q\rangle_{m_j} x_j^{m_j}}{\langle c; q\rangle_{m_{k+1}+\ldots+m_n} \prod_{j=1}^{n} \langle 1; q\rangle_{m_j}}\\
&\qquad \times {}_2\phi_1\left(a, b; c + m_{k+1} + \ldots + m_n \mid q; xq^{c-a-b+N}\right) = \mathrm{RHS}. \qquad \square
\end{aligned}
$$

Theorem 4.4. *A reduction formula for* ${}^{(k)}\Phi_{\mathrm{CD}}^{(n)}$ *and a q-analogue of* [11, p. 220 (5.3)]*:*

$$
\begin{aligned}
&{}^{(k)}\Phi_{\mathrm{CD}}^{(n)}\left[a, B, b_1, \ldots, b_k; c, c_{k+1}, \ldots, c_n \;\middle|\; q; xq^{c-a-b-N+b_2+\ldots+b_k}, \right.\\
&\qquad \left. xq^{c-a-b-N+b_3+\ldots+b_k}, \ldots, xq^{c-a-b-N}, x_{k+1}, \ldots, x_n \right]\\
&= {}^{(1)}\Phi_{\mathrm{CD}}^{(n-k+1)}\left[a, B, b; c, c_{k+1}, \ldots, c_n \;\middle|\; q; xq^{c-a-b-N}, x_{k+1}, \ldots, x_n \right].
\end{aligned}
$$

Proof.

$$
\begin{aligned}
&\mathrm{LHS} = \sum_{m_{k+1},\ldots,m_n} \frac{\langle a, B; q\rangle_{m_{k+1}+\ldots+m_n} \prod_{j=k+1}^{n} x_j^{m_j}}{\prod_{j=k+1}^{n} \langle c_j, 1; q\rangle_{m_j}}\\
&\Phi_{\mathrm{D}}^{(k)}\left[\begin{matrix} a + m_{k+1} + \ldots + m_n, b_1, \ldots, b_k \\ c \end{matrix} \;\middle|\; q; xq^{c-a-b-N+b_2+\ldots+b_k}, \right.\\
&\qquad \left. xq^{c-a-b-N+b_3+\ldots+b_k}, \ldots, xq^{c-a-b-N} \right]\\
&\overset{\text{by (7)}}{=} \sum_{m_{k+1},\ldots,m_n} \frac{\langle a, B; q\rangle_{m_{k+1}+\ldots+m_n} \prod_{j=k+1}^{n} x_j^{m_j}}{\prod_{j=k+1}^{n} \langle c_j, 1; q\rangle_{m_j}}\\
&\qquad \times {}_2\phi_1\left(a + m_{k+1} + \ldots + m_n, b; c \mid q; xq^{c-a-b-N}\right) = \mathrm{RHS}. \qquad \square
\end{aligned}
$$

There are several generating functions, we just state one. The proofs use the q-binomial theorem.

Theorem 4.5 (A q-analogue of [1, p. 204]). *Generating function for* ${}^{(k)}\Phi^{(n)}_{\mathrm{AC}}$*:*

$$\begin{aligned}
&\frac{1}{(t;q)_a}\sum_{\vec{m}}\frac{\langle a;q\rangle_m\langle b;q\rangle_{m_1+\ldots+m_k}\prod_{j=k+1}^n\langle b_j;q\rangle_{m_j}}{\prod_{j=1}^n\langle c_j;q\rangle_{m_j}\langle\vec{1};q\rangle_{\vec{m}}(tq^a;q)_m}\vec{x}^{\vec{m}}\\
&\overset{\text{by [2, (7.27)]}}{=}\sum_{r=0}^{\infty}\frac{\langle a;q\rangle_r}{\langle 1;q\rangle_r}t^r\,{}^{(k)}\Phi^{(n)}_{\mathrm{AC}}(a+r,b,b_{k+1},\ldots,b_n;c_1,\ldots,c_n\mid\\
&\qquad q;x_1,\ldots,x_n).
\end{aligned}$$

Confluent functions were given in [6]. Such confluent forms are obtained by letting parameters $\to\infty$.

Definition 9 (Confluent forms). A q-analogue of [1, p. 199]:

$$\begin{aligned}
&{}_{(1)}{}^{(k)}\Phi^{(n)}_{\mathrm{AC}}(a,b,b_{k+2},\ldots,b_n;c_1,\ldots,c_n\mid q;x_1,\ldots,x_n)\\
&\equiv\lim_{b_{k+1}\to\infty}{}^{(k)}\Phi^{(n)}_{\mathrm{AC}}(a,b,b_{k+1},\ldots,b_n;c_1,\ldots,c_n\mid q;x_1,\ldots,x_n).
\end{aligned}$$

A q-analogue of [1, p. 200]:

$$\begin{aligned}
&{}_{(2)}{}^{(k)}\Phi^{(n)}_{\mathrm{AC}}(a,b_{k+1},\ldots,b_n;c_1,\ldots,c_n\mid q;x_1,\ldots,x_n)\\
&\equiv\lim_{b\to\infty}{}^{(k)}\Phi^{(n)}_{\mathrm{AC}}(a,b,b_{k+1},\ldots,b_n;c_1,\ldots,c_n\mid q;x_1,\ldots,x_n).
\end{aligned}$$

A q-analogue of [1, p. 200]:

$$\begin{aligned}
&{}_{(1)}{}^{(k)}\Phi^{(n)}_{\mathrm{AD}}(a,b_1,\ldots,b_n;c,c_{k+2},\ldots,c_n\mid q;x_1,\ldots,x_n)\\
&\equiv\lim_{c_{k+1}\to\infty}{}^{(k)}\Phi^{(n)}_{\mathrm{AD}}(a,b_1,\ldots,b_n;c,c_{k+1},\ldots,c_n\mid q;x_1,\ldots,x_n).
\end{aligned}$$

A q-analogue of [1, p. 200]:

$$\begin{aligned}
&{}_{(1)}{}^{(k)}\Phi^{(n)}_{\mathrm{BD}}(a,a_{k+2},\ldots,a_n,b_1,\ldots,b_n;c\mid q;x_1,\ldots,x_n)\\
&\equiv\lim_{a_{k+1}\to\infty}{}^{(k)}\Phi^{(n)}_{\mathrm{BD}}(a,a_{k+1},\ldots,a_n,b_1,\ldots,b_n;c\mid q;x_1,\ldots,x_n).
\end{aligned}$$

A q-analogue of [1, p. 200]:

$$\begin{aligned}
&{}_{(2)}{}^{(k)}\Phi^{(n)}_{\mathrm{BD}}(a_{k+1},\ldots,a_n,b_1,\ldots,b_n;c\mid q;x_1,\ldots,x_n)\\
&\equiv\lim_{a\to\infty}{}^{(k)}\Phi^{(n)}_{\mathrm{BD}}(a,a_{k+1},\ldots,a_n,b_1,\ldots,b_n;c\mid q;x_1,\ldots,x_n).
\end{aligned}$$

5. Conclusion

As Karlsson pointed out, only symmetric functions give nice, short formulas. In his opinion, these functions are the best ones. In forthcoming papers, we shall follow similar paths for other functions. We have shown that our q-real numbers, in particular the third one, are useful to find convergence regions for q-hypergeometric functions.

6. Discussion

We note that in [7, (7.3)], we found a q-integral transformation formula between two intermediate q-Lauricella functions. Karlsson's intermediate Lauricella functions are not well known. More famous are the so-called Srivastava–Daoust functions. These latter functions are, however, too general, since their convergence regions are not easy to compute.

References

[1] Chandel Singh, R. C., Gupta, A. K., *Multiple hypergeometric functions related to Lauricella's functions*, Jñānābha **16** (1986), 195–209.

[2] Ernst, T., *A Comprehensive Treatment of q-Calculus*, Birkhäuser, Basel, 2012.

[3] Ernst, T., *Convergence aspects for q-Lauricella functions I*, Adv. Stud. Contemp. Math. (Kyungshang) **22** (1) (2012), 35–50.

[4] Ernst, T., *Convergence aspects for q-Appell functions I*, J. Indian Math. Soc., New Ser. **81** (1–2) (2014), 67–77.

[5] Ernst, T., *On the complex q-Appell polynomials*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **74** (1) (2020), 31–43.

[6] Ernst, T., *On confluent q-hypergeometric functions*, Appl. Anal. Optim. **4** (3) (2020), 385–404.

[7] Ernst, T., *Further results on multiple q-Eulerian integrals for various q-hypergeometric functions*, Publ. Inst. Math. (Beograd) (N.S.) **108** (122) (2020), 63–77.

[8] Ernst, T., *Three algebraic number systems based on the q-addition with applications*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **75** (2) (2021), 45–71.

[9] Exton, H., *On certain confluent hypergeometric functions of three variables*, Ganita **21** (2) (1970), 79–92.

[10] Exton, H., *Certain hypergeometric functions of four variables*, Bull. Soc. Math. Grèce (N.S.) **13** (1–2) (1972), 104–113.

[11] Karlsson, P., *On intermediate Lauricella functions*, Jñānābha **16** (1986), 211–222.

[12] Qureshi, M.I, Quraishi,K.A., Khan,B. Arora, A. *Transformations associated with quadruple hypergeometric functions of Exton and Srivastava*, Asia Pac. J. Math. **4** (1) (2017), 38–48.

[13] Srivastava, H.M., *A formal extension of certain generating functions*, Glasnik Mat. Ser. III **5** (25) (1970), 229–239.

[14] Srivastava, H.M., *A formal extension of certain generating functions II*, Glasnik Mat. Ser. III **6** (26) (1971), 35–44.

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