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On a new two-parameter generalization of dual-hyperbolic Jacobsthal numbers

Abstract. In this paper we introduce two-parameter generalization of dual-hyperbolic Jacobsthal numbers: dual-hyperbolic (s, p)-Jacobsthal numbers. We present some properties of them, among others the Binet formula, Catalan, Cassini, d'Ocagne identities. Moreover, we give the generating function, matrix generator and summation formula for these numbers.

1. Introduction. Hyperbolic and dual numbers are two dimensional number systems. A hyperbolic number is defined as $h = x + yj$, where $x, y \in \mathbb{R}$ and j is a unipotent (hyperbolic) imaginary unit such that $j^2 = 1$ and $j \neq \pm 1$. Hence the set of hyperbolic numbers is defined as

$$\mathbb{H} = \{x + yj \colon x, y \in \mathbb{R},\ j^2 = 1,\ j \neq \pm 1\}.$$

Hyperbolic imaginary unit was introduced by Cockle ([7, 8, 9, 10]). Hyperbolic numbers are well studied in the literature, see [18]. Dual numbers were introduced by Clifford ([6]). The set of dual numbers is defined in the following way:

$$\mathbb{D} = \{u + v\varepsilon \colon u, v \in \mathbb{R},\ \varepsilon^2 = 0,\ \varepsilon \neq 0\}.$$

Dual-hyperbolic numbers are known generalization of hyperbolic and dual numbers. The set of dual-hyperbolic numbers, denoted by $\mathbb{DH}$, is defined

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as follows:

$$\mathbb{DH} = \{a_1 + a_2 j + (a_3 + a_4 j)\varepsilon = a_1 + a_2 j + a_3\varepsilon + a_4 j\varepsilon \colon a_1, a_2, a_3, a_4 \in \mathbb{R}\},$$

where the base elements $(1, j, \varepsilon, j\varepsilon)$ of dual-hyperbolic numbers correspond to the following commutative multiplications:

$$j^2 = 1,\ \varepsilon^2 = (j\varepsilon)^2 = 0,\ \varepsilon(j\varepsilon) = (j\varepsilon)\varepsilon = 0,\ j(j\varepsilon) = (j\varepsilon)j = \varepsilon. \tag{1}$$

Let $w_1 = a_1 + a_2 j + a_3\varepsilon + a_4 j\varepsilon$, $w_2 = b_1 + b_2 j + b_3\varepsilon + b_4 j\varepsilon$ be any two dual-hyperbolic numbers. Then the equality, the addition, the subtraction and the multiplication by scalar are defined in the following way:

$$w_1 = w_2 \ \text{ only if } \ a_1 = b_1,\ a_2 = b_2,\ a_3 = b_3,\ a_4 = b_4,$$

$$w_1 + w_2 = a_1 + b_1 + (a_2 + b_2)j + (a_3 + b_3)\varepsilon + (a_4 + b_4)j\varepsilon,$$

$$w_1 - w_2 = a_1 - b_1 + (a_2 - b_2)j + (a_3 - b_3)\varepsilon + (a_4 - b_4)j\varepsilon,$$

$$\text{for } \ k \in \mathbb{R},\ kw_1 = ka_1 + ka_2 j + ka_3\varepsilon + ka_4 j\varepsilon.$$

Moreover, by (1) we get

$$\begin{aligned} w_1 \cdot w_2 = {} & a_1 b_1 + a_2 b_2 + (a_1 b_2 + a_2 b_1)j + (a_1 b_3 + a_2 b_4 + a_3 b_1 + a_4 b_2)\varepsilon \\ & + (a_1 b_4 + a_2 b_3 + a_3 b_2 + a_4 b_1)j\varepsilon. \end{aligned} \tag{2}$$

The dual hyperbolic numbers form a commutative ring (with identity element $1 + 0j + 0\varepsilon + 0j\varepsilon$), a real vector space and an algebra, but not every nonzero dual-hyperbolic number has an inverse, hence $(\mathbb{DH}, +, \cdot)$ is not a field. For more information on the dual-hyperbolic numbers, see [1].

2. The (s, p)-Jacobsthal numbers. Let $n \geq 0$ be an integer. The Jacobsthal sequence $\{J_n\}$ is defined by the second order linear recurrence

$$J_n = J_{n-1} + 2J_{n-2} \ \text{ for } \ n \geq 2$$

with initial terms $J_0 = 0$, $J_1 = 1$. The direct formula, named the Binet formula, for the Jacobsthal numbers has the following form:

$$J_n = \frac{2^n - (-1)^n}{3}.$$

There are many generalizations of this sequence in the literature, see [11, 12, 23]. In [2], a two-parameter generalization of the Jacobsthal numbers was introduced. We recall this generalization and some important properties of these numbers.

Let $n \geq 0$, $s \geq 0$, $p \geq 0$ be integers, sequence $\{J_n(s, p)\}$ was defined by the following recurrence:

$$J_n(s, p) = 2^{s+p} J_{n-1}(s, p) + (2^{2s+p} + 2^{s+2p}) J_{n-2}(s, p) \ \text{ for } \ n \geq 2 \tag{3}$$

with initial conditions $J_0(s, p) = 1$, $J_1(s, p) = 2^s + 2^p + 2^{s+p}$.

It is easily seen that for $s = p = 0$ we have $J_n(0, 0) = J_{n+2}$.

By (3) we obtain

$$\begin{aligned}
J_0(s,p) &= 1\\
J_1(s,p) &= 2^s + 2^p + 2^{s+p}\\
J_2(s,p) &= 2^{2s+p+1} + 2^{s+2p+1} + 2^{2s+2p}\\
J_3(s,p) &= 2^{3s+2p+1} + 2^{2s+3p+1} + 2^{3s+3p} + 2^{3s+p}\\
&\quad + 2^{2s+2p+1} + 2^{3s+2p} + 2^{2s+3p} + 2^{s+3p}.
\end{aligned}$$

Theorem 1 (Binet formula, [2]). *Let $n \geq 0$, $s \geq 0$, $p \geq 0$ be integers. Then the nth (s,p)-Jacobsthal number is given by*

$$J_n(s,p) = c_1 r_1^n + c_2 r_2^n,$$

where

$$\begin{aligned}
r_1 &= 2^{s+p-1} + \frac{1}{2}\sqrt{4^{s+p} + 2^{s+p+2}(2^s+2^p)},\\
r_2 &= 2^{s+p-1} - \frac{1}{2}\sqrt{4^{s+p} + 2^{s+p+2}(2^s+2^p)},
\end{aligned} \tag{4}$$

$$\begin{aligned}
c_1 &= \frac{2^s + 2^p + 2^{s+p} - 2^{s+p-1} + \frac{1}{2}\sqrt{4^{s+p} + 2^{s+p+2}(2^s+2^p)}}{\sqrt{4^{s+p} + 2^{s+p+2}(2^s+2^p)}},\\
c_2 &= \frac{-2^s - 2^p - 2^{s+p} + 2^{s+p-1} + \frac{1}{2}\sqrt{4^{s+p} + 2^{s+p+2}(2^s+2^p)}}{\sqrt{4^{s+p} + 2^{s+p+2}(2^s+2^p)}}.
\end{aligned} \tag{5}$$

Note that

$$r_1 + r_2 = 2^{s+p}, \tag{6}$$

$$r_1 r_2 = -(2^{2s+p} + 2^{s+2p}), \tag{7}$$

$$r_1 - r_2 = \sqrt{4^{s+p} + 2^{s+p+2}(2^s+2^p)}, \tag{8}$$

$$c_1 c_2 = \frac{-(2^s+2^p)^2}{4^{s+p} + 2^{s+p+2}(2^s+2^p)}. \tag{9}$$

Theorem 2 ([2]). *The generating function of the sequence $\{J_n(s,p)\}$ has the following form:*

$$f(x) = \frac{1 + (2^s+2^p)x}{1 - 2^{s+p}x - (2^{2s+p} + 2^{s+2p})x^2}.$$

Theorem 3 ([2]). *Let $n \geq 1$, $s \geq 0$, $p \geq 0$ be integers. Then*

$$\sum_{l=0}^{n-1} J_l(s,p) = \frac{J_n(s,p) + (2^{2s+p} + 2^{s+2p})J_{n-1}(s,p) - 1 - 2^s - 2^p}{2^{s+p}(1 + 2^s + 2^p) - 1}.$$

Theorem 4 (Convolution identity, [2]). *Let n, m, s, p be integers such that $m \geq 2$, $n \geq 1$, $s \geq 0$, $p \geq 0$. Then*

$$J_{m+n}(s,p) = 2^{s+p} J_{m-1}(s,p) J_n(s,p) + \left(2^{2s+3p} + 2^{3s+2p}\right) J_{m-2}(s,p) J_{n-1}(s,p).$$

Jacobsthal numbers and other numbers defined by the second order linear recurrence relations appear in many subjects of mathematics. These numbers have applications also in the theory of complex numbers and the theory of quaternions, see [22]. In [13], Horadam defined the Fibonacci and Lucas quaternions. In [20], [21], the authors investigated Jacobsthal and Pell quaternions. In [5], the dual-hyperbolic Fibonacci and Lucas numbers were introduced. Some generalizations of dual-hyperbolic balancing numbers were considered in [4]. In [3], a new generalization of split Jacobsthal numbers was introduced. In [19], the authors studied dual-hyperbolic Jacobsthal and Jacobsthal–Lucas numbers. Some interesting results concerning quaternions, split quaternions and hypercomplex numbers with Fibonacci coefficients were given in [14, 15, 16, 17]. Based on these ideas we define and study dual-hyperbolic (s,p)-Jacobsthal numbers.

3. Dual-hyperbolic (s,p)-Jacobsthal numbers. For a nonnegative n, s, p the nth dual-hyperbolic (s,p)-Jacobsthal number $\mathbb{DH}J_n^{s,p}$ is defined as

$$\mathbb{DH}J_n^{s,p} = J_n(s,p) + J_{n+1}(s,p)j + J_{n+2}(s,p)\varepsilon + J_{n+3}(s,p)j\varepsilon, \tag{10}$$

where $J_n(s,p)$ is given by (3).

Note that for $s = p = 0$ we obtain $\mathbb{DH}J_n^{0,0} = \mathbb{DH}J_{n+2}$, where $\mathbb{DH}J_n$ denotes nth dual-hyperbolic Jacobsthal number defined in [19].

Theorem 5. *Let $n \geq 0$, $s \geq 0$, $p \geq 0$ be integers. Then*

$$\mathbb{DH}J_{n+2}^{s,p} = 2^{s+p}\mathbb{DH}J_{n+1}^{s,p} + (2^{2s+p} + 2^{s+2p})\mathbb{DH}J_n^{s,p},$$

where

$$\begin{aligned}
\mathbb{DH}J_0^{s,p} &= 1 + (2^s + 2^p + 2^{s+p})j + (2^{2s+p+1} + 2^{s+2p+1} + 2^{2s+2p})\varepsilon \\
&\quad + (2^{3s+2p+1} + 2^{2s+3p+1} + 2^{3s+3p} + 2^{3s+p} \\
&\quad + 2^{2s+2p+1} + 2^{3s+2p} + 2^{2s+3p} + 2^{s+3p})j\varepsilon, \\
\mathbb{DH}J_1^{s,p} &= 2^s + 2^p + 2^{s+p} + (2^{2s+p+1} + 2^{s+2p+1} + 2^{2s+2p})j \\
&\quad + (2^{3s+2p+1} + 2^{2s+3p+1} + 2^{3s+3p} + 2^{3s+p} \\
&\quad + 2^{2s+2p+1} + 2^{3s+2p} + 2^{2s+3p} + 2^{s+3p})\varepsilon \\
&\quad + (2^{4s+3p+1} + 2^{3s+4p+2} + 2^{4s+4p} + 2^{4s+3p+1} \\
&\quad + 3 \cdot 2^{2s+4p} + 3 \cdot 2^{4s+2p} + 3 \cdot 2^{3s+3p+1})j\varepsilon.
\end{aligned}$$

Proof. By formula (10) we have

$$\begin{aligned}
&2^{s+p}\mathbb{DH}J^{s,p}_{n+1} + (2^{2s+p} + 2^{s+2p})\mathbb{DH}J^{s,p}_{n}\\
&= 2^{s+p}(J_{n+1}(s,p) + J_{n+2}(s,p)j + J_{n+3}(s,p)\varepsilon + J_{n+4}(s,p)j\varepsilon)\\
&\quad + (2^{2s+p} + 2^{s+2p})(J_n(s,p) + J_{n+1}(s,p)j + J_{n+2}(s,p)\varepsilon + J_{n+3}(s,p)j\varepsilon)\\
&= J_{n+2}(s,p) + J_{n+3}(s,p)j + J_{n+4}(s,p)\varepsilon + J_{n+5}(s,p)j\varepsilon = \mathbb{DH}J^{s,p}_{n+2}.
\end{aligned}$$

□

Theorem 6. *Let $n \geq 0$, $s \geq 0$, $p \geq 0$ be integers. Then*

$$\mathbb{DH}J^{s,p}_{n} - j\mathbb{DH}J^{s,p}_{n+1} - \varepsilon\mathbb{DH}J^{s,p}_{n+2} + j\varepsilon\mathbb{DH}J^{s,p}_{n+3} = J_n(s,p) - J_{n+2}(s,p).$$

Proof. By simple calculations we get

$$\begin{aligned}
&\mathbb{DH}J^{s,p}_{n} - j\mathbb{DH}J^{s,p}_{n+1} - \varepsilon\mathbb{DH}J^{s,p}_{n+2} + j\varepsilon\mathbb{DH}J^{s,p}_{n+3}\\
&= J_n(s,p) + J_{n+1}(s,p)j + J_{n+2}(s,p)\varepsilon + J_{n+3}(s,p)j\varepsilon\\
&\quad - j\left(J_{n+1}(s,p) + J_{n+2}(s,p)j + J_{n+3}(s,p)\varepsilon + J_{n+4}(s,p)j\varepsilon\right)\\
&\quad - \varepsilon\left(J_{n+2}(s,p) + J_{n+3}(s,p)j + J_{n+4}(s,p)\varepsilon + J_{n+5}(s,p)j\varepsilon\right)\\
&\quad + j\varepsilon\left(J_{n+3}(s,p) + J_{n+4}(s,p)j + J_{n+5}(s,p)\varepsilon + J_{n+6}(s,p)j\varepsilon\right)\\
&= J_n(s,p) + J_{n+1}(s,p)j + J_{n+2}(s,p)\varepsilon + J_{n+3}(s,p)j\varepsilon\\
&\quad - J_{n+1}(s,p)j - J_{n+2}(s,p) - J_{n+3}(s,p)j\varepsilon - J_{n+4}(s,p)\varepsilon\\
&\quad - J_{n+2}(s,p)\varepsilon - J_{n+3}(s,p)j\varepsilon + J_{n+3}(s,p)j\varepsilon + J_{n+4}(s,p)\varepsilon\\
&= J_n(s,p) - J_{n+2}(s,p).
\end{aligned}$$

□

Theorem 7 (Binet type formula). *Let $n \geq 0$, $s \geq 0$, $p \geq 0$ be integers. Then*

$$\mathbb{DH}J^{s,p}_{n} = c_1\hat{r_1}r_1^n + c_2\hat{r_2}r_2^n, \tag{11}$$

where

$$\hat{r_1} = 1 + r_1j + r_1^2\varepsilon + r_1^3j\varepsilon, \quad \hat{r_2} = 1 + r_2j + r_2^2\varepsilon + r_2^3j\varepsilon$$

and r_1, r_2, c_1, c_2 are given by (4), (5), respectively.

Proof. By Theorem 1 we get

$$\begin{aligned}
\mathbb{DH}J^{s,p}_{n} &= J_n(s,p) + J_{n+1}(s,p)j + J_{n+2}(s,p)\varepsilon + J_{n+3}(s,p)j\varepsilon\\
&= c_1r_1^n + c_2r_2^n + \left(c_1r_1^{n+1} + c_2r_2^{n+1}\right)j\\
&\quad + \left(c_1r_1^{n+2} + c_2r_2^{n+2}\right)\varepsilon + \left(c_1r_1^{n+3} + c_2r_2^{n+3}\right)j\varepsilon\\
&= c_1r_1^n(1 + r_1j + r_1^2\varepsilon + r_1^3j\varepsilon) + c_2r_2^n(1 + r_2j + r_2^2\varepsilon + r_2^3j\varepsilon)\\
&= c_1\hat{r_1}r_1^n + c_2\hat{r_2}r_2^n.
\end{aligned}$$

□

Corollary 8 (Binet type formula for dual-hyperbolic Jacobsthal numbers). *Let $n \geq 0$ be an integer. Then*

$$\mathbb{DH}J_n = \frac{1}{3}\left[2^n(1 + 2j + 4\varepsilon + 8j\varepsilon) - (-1)^n(1 - j + \varepsilon - j\varepsilon)\right].$$

Proof. By formula (11), for $s=p=0$ we have $c_1=\frac{4}{3}$, $c_2=-\frac{1}{3}$, $r_1=2$, $r_2=-1$ and

$$\begin{aligned}\mathbb{DH}J_n^{0,0} &= \frac{4}{3}\cdot 2^n(1+2j+4\varepsilon+8j\varepsilon)-\frac{1}{3}(-1)^n(1-j+\varepsilon-j\varepsilon)\\ &= \frac{1}{3}\cdot 2^{n+2}(1+2j+4\varepsilon+8j\varepsilon)-\frac{1}{3}(-1)^{n+2}(1-j+\varepsilon-j\varepsilon)\\ &= \mathbb{DH}J_{n+2}.\end{aligned}$$

□

By simple calculations we get

$$\begin{aligned}\hat{r_1}\hat{r_2}=\hat{r_2}\hat{r_1} &= (1+r_1j+r_1^2\varepsilon+r_1^3j\varepsilon)(1+r_2j+r_2^2\varepsilon+r_2^3j\varepsilon)\\ &= 1+r_1r_2+(r_1+r_2)j+(r_1^2+r_2^2)(1+r_1r_2)\varepsilon\\ &\quad+(r_1^3+r_2^3+r_1r_2(r_1+r_2))j\varepsilon.\end{aligned}$$

Using the equalities (6), (7), we have

$$\begin{aligned}r_1^2+r_2^2 &= (r_1+r_2)^2-2r_1r_2=4^{s+p}+2^{2s+p+1}+2^{s+2p+1},\\ r_1^3+r_2^3 &= (r_1+r_2)^3-3r_1r_2(r_1+r_2)=8^{s+p}+3(2^{3s+2p}+2^{2s+3p}).\end{aligned}$$

Hence

$$\begin{aligned}\hat{r_1}\hat{r_2} &= 1-2^{2s+p}-2^{s+2p}+2^{s+p}j\\ &\quad+(2^{2s+2p}-2^{4s+3p}-2^{3s+4p}+2^{2s+p+1}\\ &\quad-2^{4s+2p+1}+2^{s+2p+1}-2^{3s+3p+2}-2^{2s+4p+1})\varepsilon\\ &\quad+(8^{s+p}+2^{3s+2p+1}+2^{2s+3p+1})j\varepsilon.\end{aligned} \tag{12}$$

The following theorem gives the d'Ocagne type identity involving the dual-hyperbolic (s,p)-Jacobsthal numbers.

Theorem 9 (d'Ocagne type identity for dual-hyperbolic (s,p)-Jacobsthal numbers). *Let $n\geq 0$, $m\geq 0$, $s\geq 0$, $p\geq 0$ be integers such that $m\geq n$. Then*

$$\begin{aligned}&\mathbb{DH}J_m^{s,p}\cdot\mathbb{DH}J_{n+1}^{s,p}-\mathbb{DH}J_{m+1}^{s,p}\cdot\mathbb{DH}J_n^{s,p}\\ &=\frac{-(2^s+2^p)^2(-1)^n(2^{2s+p}+2^{s+2p})^n}{\sqrt{4^{s+p}+2^{s+p+2}(2^s+2^p)}}\hat{r_1}\hat{r_2}(r_2^{m-n}-r_1^{m-n}),\end{aligned}$$

where r_1, r_2, $\hat{r_1}\hat{r_2}$ are given by (4), (12)*, respectively.*

Proof. By formula (11) we get

$$\begin{aligned}&\mathbb{DH}J_m^{s,p}\cdot\mathbb{DH}J_{n+1}^{s,p}-\mathbb{DH}J_{m+1}^{s,p}\cdot\mathbb{DH}J_n^{s,p}\\ &=(c_1\hat{r_1}r_1^m+c_2\hat{r_2}r_2^m)(c_1\hat{r_1}r_1^{n+1}+c_2\hat{r_2}r_2^{n+1})\\ &\quad-(c_1\hat{r_1}r_1^{m+1}+c_2\hat{r_2}r_2^{m+1})(c_1\hat{r_1}r_1^n+c_2\hat{r_2}r_2^n)\\ &=c_1c_2\hat{r_1}\hat{r_2}\left(r_2^mr_1^{n+1}+r_1^mr_2^{n+1}-r_2^{m+1}r_1^n-r_1^{m+1}r_2^n\right)\\ &=c_1c_2\hat{r_1}\hat{r_2}(r_1r_2)^n(r_1-r_2)(r_2^{m-n}-r_1^{m-n}).\end{aligned}$$

Using formulas (7), (8) and (9), we get

$$\begin{aligned}&\mathbb{DH}J_m^{s,p}\cdot\mathbb{DH}J_{n+1}^{s,p}-\mathbb{DH}J_{m+1}^{s,p}\cdot\mathbb{DH}J_n^{s,p}\\&=\frac{-(2^s+2^p)^2(-1)^n(2^{2s+p}+2^{s+2p})^n}{\sqrt{4^{s+p}+2^{s+p+2}(2^s+2^p)}}\hat{r_1}\hat{r_2}(r_2^{m-n}-r_1^{m-n}).\end{aligned}$$ □

Corollary 10 (d'Ocagne type identity for dual-hyperbolic Jacobsthal numbers). *Let $n\geq 0$, $m\geq 0$ be integers such that $m\geq n$. Then*

$$\begin{aligned}&\mathbb{DH}J_m\cdot\mathbb{DH}J_{n+1}-\mathbb{DH}J_{m+1}\cdot\mathbb{DH}J_n\\&=\frac{4}{3}(-2)^n\left(2^{m-n}-(-1)^{m-n}\right)(-1+j-5\varepsilon+5j\varepsilon).\end{aligned}$$

Theorem 11 (Catalan identity for dual-hyperbolic (s,p)-Jacobsthal numbers). *Let $n\geq 0$, $m\geq 0$, $s\geq 0$, $p\geq 0$ be integers such that $n\geq m$. Then*

$$\begin{aligned}&(\mathbb{DH}J_n^{s,p})^2-\mathbb{DH}J_{n-m}^{s,p}\cdot\mathbb{DH}J_{n+m}^{s,p}\\&=\frac{(2^s+2^p)^2}{4^{s+p}+2^{s+p+2}(2^s+2^p)}(-(2^{s+2p}+2^{s+2p}))^{n-m}\hat{r_1}\hat{r_2}\left({r_1}^m-r_2^m\right)^2,\end{aligned}$$

where r_1, r_2, $\hat{r_1}\hat{r_2}$ are given by (4), (12), respectively.

Proof. By formula (11) we get

$$\begin{aligned}&(\mathbb{DH}J_n^{s,p})^2-\mathbb{DH}J_{n-m}^{s,p}\cdot\mathbb{DH}J_{n+m}^{s,p}\\&=(c_1\hat{r_1}r_1^n+c_2\hat{r_2}r_2^n)(\hat{r_1}r_1^n+c_2\hat{r_2}r_2^n)\\&\quad-(c_1\hat{r_1}r_1^{n-m}+c_2\hat{r_2}r_2^{n-m})(c_1\hat{r_1}r_1^{n+m}+c_2\hat{r_2}r_2^{n+m})\\&=2c_1c_2\hat{r_1}\hat{r_2}(r_1r_2)^n-c_1c_2\hat{r_1}\hat{r_2}r_1^{n+m}r_2^{n-m}-c_1c_2\hat{r_1}\hat{r_2}r_1^{n-m}r_2^{n+m}\\&=c_1c_2\hat{r_1}\hat{r_2}(r_1r_2)^n\left(2-\left(\frac{r_1}{r_2}\right)^m-\left(\frac{r_2}{r_1}\right)^m\right)\\&=-c_1c_2\hat{r_1}\hat{r_2}(r_1r_2)^{n-m}(r_1^m-r_2^m)^2.\end{aligned}$$

Using formulas (7) and (9), we obtain the result. □

For $m=1$ we obtain Cassini type identity for the dual-hyperbolic (s,p)-Jacobsthal numbers.

Corollary 12 (Cassini type identity for dual-hyperbolic (s,p)-Jacobsthal numbers). *Let $n\geq 1$, $s\geq 0$, $p\geq 0$ be integers. Then*

$$(\mathbb{DH}J_n^{s,p})^2-\mathbb{DH}J_{n-1}^{s,p}\cdot\mathbb{DH}J_{n+1}^{s,p}=(2^s+2^p)^2(-(2^{s+2p}+2^{s+2p}))^{n-1}\hat{r_1}\hat{r_2},$$

where $\hat{r_1}\hat{r_2}$ is given by (12).

In particular, by Theorem 11, we obtain the following formulas for the dual-hyperbolic Jacobsthal numbers.

Corollary 13 (Catalan type identity for dual-hyperbolic Jacobsthal numbers). *Let $n \geq 0$, $m \geq 0$, be integers such that $n \geq m$. Then*

$$(\mathbb{DH}J_n)^2 - \mathbb{DH}J_{n-m} \cdot \mathbb{DH}J_{n+m} = \frac{4}{9}(-2)^{n-m}(2^m - (-1)^m))^2(-1+j-5\varepsilon+5j\varepsilon).$$

Corollary 14 (Cassini type identity for dual-hyperbolic Jacobsthal numbers). *Let $n \geq 1$ be an integer. Then*

$$(\mathbb{DH}J_n)^2 - \mathbb{DH}J_{n-1} \cdot \mathbb{DH}J_{n+1} = 4(-2)^{n-1}(-1+j-5\varepsilon+5j\varepsilon).$$

The next theorem presents the convolution identity for the dual-hyperbolic (s,p)-Jacobsthal numbers.

Theorem 15. *Let n, m, s, p be integers such that $m \geq 2$, $n \geq 1$, $s \geq 0$, $p \geq 0$. Then*

$$\begin{aligned} 2\mathbb{DH}J^{s,p}_{m+n} &= 2^{s+p}\mathbb{DH}J^{s,p}_{m-1} \cdot \mathbb{DH}J^{s,p}_{n} + (2^{2s+3p} + 2^{3s+2p})\mathbb{DH}J^{s,p}_{m-2} \cdot \mathbb{DH}J^{s,p}_{n-1} \\ &\quad + J_{m+n}(s,p) - J_{m+n+2}(s,p) - 2J_{m+n+4}(s,p)\varepsilon - 2J_{m+n+3}(s,p)j\varepsilon. \end{aligned}$$

Proof. Let $A = 2^{s+p}$, $B = 2^{2s+3p} + 2^{3s+2p}$. Then by simple calculations, using (2), we have

$$\begin{aligned} &A \cdot \mathbb{DH}J^{s,p}_{m-1} \cdot \mathbb{DH}J^{s,p}_{n} + B \cdot \mathbb{DH}J^{s,p}_{m-2} \cdot \mathbb{DH}J^{s,p}_{n-1} \\ &\quad = A[J_{m-1}(s,p)J_n(s,p) + J_{m-1}(s,p)J_{n+1}(s,p)j \\ &\qquad + J_{m-1}(s,p)J_{n+2}(s,p)\varepsilon + J_{m-1}(s,p)J_{n+3}(s,p)j\varepsilon \\ &\qquad + J_m(s,p)J_n(s,p)j + J_m(s,p)J_{n+1}(s,p) + J_m(s,p)J_{n+2}(s,p)j\varepsilon \\ &\qquad + J_m(s,p)J_{n+3}(s,p)\varepsilon + J_{m+1}(s,p)J_n(s,p)\varepsilon + J_{m+1}(s,p)J_{n+1}(s,p)j\varepsilon \\ &\qquad + J_{m+2}(s,p)J_n(s,p)j\varepsilon + J_{m+2}(s,p)J_{n+1}(s,p)\varepsilon] \\ &\qquad + B[J_{m-2}(s,p)J_{n-1}(s,p) + J_{m-2}(s,p)J_n(s,p)j \\ &\qquad + J_{m-2}(s,p)J_{n+1}(s,p)\varepsilon + J_{m-2}(s,p)J_{n+2}(s,p)j\varepsilon \\ &\qquad + J_{m-1}(s,p)J_{n-1}(s,p)j + J_{m-1}(s,p)J_n(s,p) \\ &\qquad + J_{m-1}(s,p)J_{n+1}(s,p)j\varepsilon + J_{m-1}(s,p)J_{n+2}(s,p)\varepsilon \\ &\qquad + J_m(s,p)J_{n-1}(s,p)\varepsilon + J_m(s,p)J_n(s,p)j\varepsilon \\ &\qquad + J_{m+1}(s,p)J_{n-1}(s,p)j\varepsilon + J_{m+1}(s,p)J_n(s,p)\varepsilon]. \end{aligned}$$

By simple calculations we get

$$\begin{aligned} &A \cdot \mathbb{DH}J^{s,p}_{m-1} \cdot \mathbb{DH}J^{s,p}_{n} + B \cdot \mathbb{DH}J^{s,p}_{m-2} \cdot \mathbb{DH}J^{s,p}_{n-1} \\ &\quad = A\,J_{m-1}(s,p)J_n(s,p) + B\,J_{m-2}(s,p)J_{n-1}(s,p) \\ &\qquad + [A\,J_{m-1}(s,p)J_{n+1}(s,p) + B\,J_{m-2}(s,p)J_n(s,p) \\ &\qquad + A\,J_m(s,p)J_n(s,p) + B\,J_{m-1}(s,p)J_{n-1}(s,p)]j \\ &\qquad + [A\,J_{m-1}(s,p)J_{n+2}(s,p) + B\,J_{m-2}(s,p)J_{n+1}(s,p) \\ &\qquad + A\,J_{m+1}(s,p)J_n(s,p) + B\,J_m(s,p)J_{n-1}(s,p)]\varepsilon \end{aligned}$$

$$
\begin{aligned}
&+ [A\, J_{m-1}(s,p)J_{n+3}(s,p) + B\, J_{m-2}(s,p)J_{n+2}(s,p)\\
&+ A\, J_m(s,p)J_{n+2}(s,p) + B\, J_{m-1}(s,p)J_{n+1}(s,p)]j\varepsilon\\
&+ A\, J_m(s,p)J_{n+1}(s,p) + B\, J_{m-1}(s,p)J_n(s,p)\\
&+ [A\, J_m(s,p)J_{n+3}(s,p) + B\, J_{m-1}(s,p)J_{n+2}(s,p)\\
&+ A\, J_{m+2}(s,p)J_{n+1}(s,p) + B\, J_{m+1}(s,p)J_n(s,p)]\varepsilon\\
&+ [A\, J_{m+1}(s,p)J_{n+1}(s,p) + B\, J_m(s,p)J_n(s,p)\\
&+ A\, J_{m+2}(s,p)J(r,n) + B\, J_{m+1}(s,p)J_{n-1}(s,p)]j\varepsilon.
\end{aligned}
$$

Using Theorem 4, we obtain

$$
\begin{aligned}
&A\cdot \mathbb{DH}J^{s,p}_{m-1}\cdot \mathbb{DH}J^{s,p}_{n} + B\cdot \mathbb{DH}J^{s,p}_{m-2}\cdot \mathbb{DH}J^{s,p}_{n-1}\\
&\quad = J_{m+n}(s,p) + 2[J_{m+n+1}(s,p)j + J_{m+n+2}(s,p)\varepsilon + J_{m+n+3}(s,p)j\varepsilon]\\
&\qquad + J_{m+n+2}(s,p) + 2J_{m+n+4}(s,p)\varepsilon + 2J_{m+n+3}(s,p)j\varepsilon\\
&\quad = 2\mathbb{DH}J^{s,p}_{m+n} - J_{m+n}(s,p) + J_{m+n+2}(s,p)\\
&\qquad + 2J_{m+n+4}(s,p)\varepsilon + 2J_{m+n+3}(s,p)j\varepsilon.
\end{aligned}
$$

Hence we get the result. □

Theorem 16. *Let $n \geq 0$, $s \geq 0$, $p \geq 0$ be integers. Then*

$$
\begin{aligned}
&\sum_{l=0}^{n} \mathbb{DH}J^{s,p}_{l}\\
&\quad = \frac{\mathbb{DH}J^{s,p}_{n+1} + (2^{2s+p} + 2^{s+2p})\mathbb{DH}J^{s,p}_{n} - (1+2^s+2^p)(1+j+\varepsilon+j\varepsilon)}{2^{s+p}(1+2^s+2^p)-1}\\
&\qquad - j - (1+2^s+2^p+2^{s+p})\varepsilon\\
&\qquad - (1+2^s+2^p+2^{s+p}+2^{2s+p+1}+2^{s+2p+1}+2^{2s+2p})j\varepsilon.
\end{aligned}
$$

Proof. By (10) we have

$$
\begin{aligned}
&\sum_{l=0}^{n} \mathbb{DH}J^{s,p}_{l} = \mathbb{DH}J^{s,p}_{0} + \mathbb{DH}J^{s,p}_{1} + \ldots + \mathbb{DH}J^{s,p}_{n}\\
&\quad = J_0(s,p) + J_1(s,p)j + J_2(s,p)\varepsilon + J_3(s,p)j\varepsilon\\
&\qquad + J_1(s,p) + J_2(s,p)j + J_3(s,p)\varepsilon + J_4(s,p)j\varepsilon + \ldots\\
&\qquad + J_n(s,p) + J_{n+1}(s,p)j + J_{n+2}(s,p)\varepsilon + J_{n+3}(s,p)j\varepsilon\\
&\quad = J_0(s,p) + J_1(s,p) + \ldots + J_n(s,p)\\
&\qquad + (J_1(s,p) + J_2(s,p) + \ldots + J_{n+1}(s,p) + J_0(s,p) - J_0(s,p))\, j+\\
&\qquad + (J_2(s,p) + J_3(s,p) + \ldots + J_{n+2}(s,p) + J_0(s,p) + J_1(s,p)\\
&\qquad - J_0(s,p) - J_1(s,p))\varepsilon\\
&\qquad + (J_3(s,p) + J_4(s,p) + \ldots + J_{n+3}(s,p) + J_0(s,p) + J_1(s,p) + J_2(s,p)\\
&\qquad - J_0(s,p) - J_1(s,p) - J_2(s,p))j\varepsilon.
\end{aligned}
$$

Using Theorem 3, we obtain

$$\sum_{l=0}^{n}\mathbb{DH}J_l^{s,p}\frac{1}{2^{s+p}(1+2^s+2^p)-1}[J_{n+1}(s,p)+(2^{2s+p}+2^{s+2p})J_n(s,p)$$
$$-1-2^s-2^p+(J_{n+2}(s,p)+(2^{2s+p}+2^{s+2p})J_{n+1}(s,p)-1-2^s-2^p)j$$
$$+(J_{n+3}(s,p)+(2^{2s+p}+2^{s+2p})J_{n+2}(s,p)-1-2^s-2^p)\varepsilon$$
$$+(J_{n+4}(s,p)+(2^{2s+p}+2^{s+2p})J_{n+3}(s,p)-1-2^s-2^p))j\varepsilon]$$
$$-(J_0(s,p)j+(J_0(s,p)+J_1(s,p))\varepsilon+(J_0(s,p)+J_1(s,p)+J_2(s,p))j\varepsilon)$$
$$=\frac{1}{2^{s+p}(1+2^s+2^p)-1}[J_{n+1}(s,p)+J_{n+2}(s,p)j+J_{n+3}(s,p)\varepsilon+J_{n+4}(s,p)j\varepsilon$$
$$+(2^{2s+p}+2^{s+2p})(J_n(s,p)+J_{n+1}(s,p)j+J_{n+2}(s,p)\varepsilon+J_{n+3}(s,p)j\varepsilon)$$
$$-(1+2^s+2^p)(1+j+\varepsilon+j\varepsilon)]-j-(1+2^s+2^p+2^{s+p})\varepsilon$$
$$-(1+2^s+2^p+2^{s+p}+2^{2s+p+1}+2^{s+2p+1}+2^{2s+2p})j\varepsilon$$
$$=\frac{\mathbb{DH}J_{n+1}^{s,p}+(2^{2s+p}+2^{s+2p})\mathbb{DH}J_n^{s,p}-(1+2^s+2^p)(1+j+\varepsilon+j\varepsilon)}{2^{s+p}(1+2^s+2^p)-1}$$
$$-j-(1+2^s+2^p+2^{s+p})\varepsilon$$
$$-(1+2^s+2^p+2^{s+p}+2^{2s+p+1}+2^{s+2p+1}+2^{2s+2p})j\varepsilon. \qquad \square$$

In particular, we obtain the following formula for the dual-hyperbolic Jacobsthal numbers.

Corollary 17. *Let $n \geq 1$ be an integer. Then*

$$\sum_{l=0}^{n}\mathbb{DH}J_l=\frac{\mathbb{DH}J_{n+2}-\mathbb{DH}J_1}{2}.$$

Proof. By Theorem 16 for $s=p=0$ we have

$$\sum_{l=0}^{n}\mathbb{DH}J_l^{0,0}=\frac{\mathbb{DH}J_{n+1}^{0,0}+2\mathbb{DH}J_n^{0,0}-3(1+j+\varepsilon+j\varepsilon)}{2}-(j+4\varepsilon+9j\varepsilon)$$
$$=\frac{\mathbb{DH}J_{n+2}^{0,0}-(3+5j+11\varepsilon+21j\varepsilon)}{2}.$$

Using formulas $J_n(0,0)=J_{n+2}$ and $\mathbb{DH}J_0=j+\varepsilon+3j\varepsilon$, $\mathbb{DH}J_1=1+j+3\varepsilon+5j\varepsilon$, we get

$$\sum_{l=0}^{n}\mathbb{DH}J_l=\frac{\mathbb{DH}J_{n+2}-(3+5j+11\varepsilon+21j\varepsilon)}{2}+\mathbb{DH}J_0+\mathbb{DH}J_1$$
$$=\frac{\mathbb{DH}J_{n+2}-(3+5j+11\varepsilon+21j\varepsilon)+2(1+2j+4\varepsilon+8j\varepsilon)}{2}$$
$$=\frac{\mathbb{DH}J_{n+2}-(1+j+3\varepsilon+5j\varepsilon)}{2}=\frac{\mathbb{DH}J_{n+2}-\mathbb{DH}J_1}{2},$$

which ends the proof. □

Now we give the ordinary generating functions for the dual-hyperbolic (s,p)-Jacobsthal numbers.

Theorem 18. *The generating function for the dual-hyperbolic (s,p)-Jacobsthal sequence $\{\mathbb{DH}J_n^{s,p}\}$ has the following form:*

$$g(x) = \frac{\mathbb{DH}J_0^{s,p} + (\mathbb{DH}J_1^{s,p} - 2^{s+p}\mathbb{DH}J_0^{s,p})x}{1 - 2^{s+p}x - (2^{2s+p} + 2^{s+2p})x^2}.$$

Proof. Assuming that the generating function of the dual-hyperbolic (s,p)-Jacobsthal sequence $\{\mathbb{DH}J_n^{s,p}\}$ has the form $g(x) = \sum_{n=0}^{\infty} \mathbb{DH}J_n^{s,p}x^n$, we obtain

$$\begin{aligned}
&(1 - 2^{s+p}x - (2^{2s+p} + 2^{s+2p})x^2)g(x)\\
&= (1 - 2^{s+p}x - (2^{2s+p} + 2^{s+2p})x^2) \cdot (\mathbb{DH}J_0^{s,p} + \mathbb{DH}J_1^{s,p}x + \mathbb{DH}J_2^{s,p}x^2 + \cdots)\\
&= \mathbb{DH}J_0^{s,p} + \mathbb{DH}J_1^{s,p}x + \mathbb{DH}J_2^{s,p}x^2 + \cdots\\
&\quad - 2^{s+p}\mathbb{DH}J_0^{s,p}x - 2^{s+p}\mathbb{DH}J_1^{s,p}x^2 - 2^{s+p}\mathbb{DH}J_2^{s,p}x^3 - \cdots\\
&\quad - (2^{2s+p} + 2^{s+2p})\mathbb{DH}J_0^{s,p}x^2 - (2^{2s+p} + 2^{s+2p})\mathbb{DH}J_1^{s,p}x^3\\
&\quad - (2^{2s+p} + 2^{s+2p})\mathbb{DH}J_2^{s,p}x^4 - \cdots\\
&= \mathbb{DH}J_0^{s,p} + (\mathbb{DH}J_1^{s,p} - 2^{s+p}\mathbb{DH}J_0^{s,p})x,
\end{aligned}$$

since $\mathbb{DH}J_n^{s,p} = 2^{s+p}\mathbb{DH}J_{n-1}^{s,p} + (2^{2s+p} + 2^{s+2p})\mathbb{DH}J_{n-2}^{s,p}$ and the coefficients of x^n for $n \geq 2$ are equal to zero. □

At the end, we give the matrix representation of the dual-hyperbolic (s,p)-Jacobsthal numbers.

Let

$$\mathbb{DH}J^{s,p}(n) = \begin{bmatrix} \mathbb{DH}J_{n+1}^{s,p} & \mathbb{DH}J_n^{s,p} \\ \mathbb{DH}J_n^{s,p} & \mathbb{DH}J_{n-1}^{s,p} \end{bmatrix}$$

be a matrix with entries being dual-hyperbolic (s,p)-Jacobsthal numbers.

Theorem 19. *Let $n \geq 1$, $s \geq 0$, $p \geq 0$ be integers. Then*

$$\begin{bmatrix} \mathbb{DH}J_{n+1}^{s,p} & \mathbb{DH}J_n^{s,p} \\ \mathbb{DH}J_n^{s,p} & \mathbb{DH}J_{n-1}^{s,p} \end{bmatrix} = \begin{bmatrix} \mathbb{DH}J_2^{s,p} & \mathbb{DH}J_1^{s,p} \\ \mathbb{DH}J_1^{s,p} & \mathbb{DH}J_0^{s,p} \end{bmatrix} \cdot \begin{bmatrix} 2^{s+p} & 1 \\ 2^{2s+p} + 2^{s+2p} & 0 \end{bmatrix}^{n-1}. \tag{13}$$

Proof. (by induction on n) It is easily seen that for $n = 1$ the result follows. Assume that formula (13) is true for $n \geq 1$. We will prove that

$$\begin{bmatrix} \mathbb{DH}J_{n+2}^{s,p} & \mathbb{DH}J_{n+1}^{s,p} \\ \mathbb{DH}J_{n+1}^{s,p} & \mathbb{DH}J_n^{s,p} \end{bmatrix} = \begin{bmatrix} \mathbb{DH}J_2^{s,p} & \mathbb{DH}J_1^{s,p} \\ \mathbb{DH}J_1^{s,p} & \mathbb{DH}J_0^{s,p} \end{bmatrix} \cdot \begin{bmatrix} 2^{s+p} & 1 \\ 2^{2s+p} + 2^{s+2p} & 0 \end{bmatrix}^{n}.$$

By induction's hypothesis we have

$$\begin{bmatrix} \mathbb{DH}J_2^{s,p} & \mathbb{DH}J_1^{s,p} \\ \mathbb{DH}J_1^{s,p} & \mathbb{DH}J_0^{s,p} \end{bmatrix} \cdot \begin{bmatrix} 2^{s+p} & 1 \\ 2^{2s+p}+2^{s+2p} & 0 \end{bmatrix}^{n-1} \cdot \begin{bmatrix} 2^{s+p} & 1 \\ 2^{2s+p}+2^{s+2p} & 0 \end{bmatrix}$$
$$= \begin{bmatrix} \mathbb{DH}J_{n+1}^{s,p} & \mathbb{DH}J_n^{s,p} \\ \mathbb{DH}J_n^{s,p} & \mathbb{DH}J_{n-1}^{s,p} \end{bmatrix} \cdot \begin{bmatrix} 2^{s+p} & 1 \\ 2^{2s+p}+2^{s+2p} & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 2^{s+p}\mathbb{DH}J_{n+1}^{s,p} + (2^{2s+p}+2^{s+2p})\mathbb{DH}J_n^{s,p} & \mathbb{DH}J_{n+1}^{s,p} \\ 2^{s+p}\mathbb{DH}J_n^{s,p} + (2^{2s+p}+2^{s+2p})\mathbb{DH}J_{n-1}^{s,p} & \mathbb{DH}J_n^{s,p} \end{bmatrix}$$
$$= \begin{bmatrix} \mathbb{DH}J_{n+2}^{s,p} & \mathbb{DH}J_{n+1}^{s,p} \\ \mathbb{DH}J_{n+1}^{s,p} & \mathbb{DH}J_n^{s,p} \end{bmatrix},$$

which ends the proof. □

Calculating the determinants in formula (13), we obtain the Cassini identity for the dual-hyperbolic (s,p)-Jacobsthal numbers.

Corollary 20. *Let $n \geq 1$, $s \geq 0$, $p \geq 0$ be integers. Then*

$$\mathbb{DH}J_{n+1}^{s,p} \cdot \mathbb{DH}J_{n-1}^{s,p} - (\mathbb{DH}J_n^{s,p})^2$$
$$= \left(\mathbb{DH}J_2^{s,p} \cdot \mathbb{DH}J_0^{s,p} - (\mathbb{DH}J_1^{s,p})^2\right)(-1)^{n-1}(2^{2s+p}+2^{s+2p})^{n-1}.$$

4. Conclusion. In this study, a two-parameter generalization of the dual-hyperbolic Jacobsthal numbers was introduced. Some results including the Binet formula, generating function, a summation formula for these numbers were presented. Moreover, some identities, such as Catalan, Cassini, d'Ocagne and convolution identities, involving the dual-hyperbolic (s,p)-Jacobsthal numbers were given. The obtained results are generalization of the results for the dual-hyperbolic Jacobsthal numbers.

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