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KAZIMIERZ GOEBEL

Remarks on retracting balls on spherical caps in c_0, c, l^∞ spaces

Abstract. For any infinite dimensional Banach space there exists a lipschitzian retraction of the closed unit ball B onto the unit sphere S. Lipschitz constants for such retractions are, in general, only roughly estimated. The paper is illustrative. It contains remarks, illustrations and estimates concerning optimal retractions onto spherical caps for sequence spaces with the uniform norm.

1. Introduction. Let X be a Banach space with the norm $\|\cdot\|$, the closed unit ball B and unit sphere S. If $\dim X$ is finite, in consequence to Brouwer's fixed point theorem, S is not the *retract* of B. It means that there are no continuous mappings $R : B \to S$ such that $x = Sx$ for all $x \in S$.

Since the work of Benyamini and Sternfeld ([3], 1983) it is known that the above fails in the infinitely dimensional Banach spaces in a strong sense. Due to the result, $\dim X = \infty$ implies the existence of a *retraction* $R : B \to S$ satisfying on B the Lipschitz condition

$$\|Rx - Ry\| \leq k \|x - y\| \tag{1.1}$$

with a certain constant $k > 0$ (being k-lipschitzian). The smallest k for which (1.1) holds is said to be the Lipschitz constant of R and is denoted by $k(R)$. The same convention will be used for other lipschitzian mappings.

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The research on the optimal retraction problem deals with estimating the *optimal retraction constant*

$$k_0(X) = \inf\left[k : \text{there exists a retraction } R : B \to S \text{ satisfying (1.1)}\right].$$

Exact value of $k_0(X)$ is not known for any space X. Various constructions and estimates for a number of spaces can be found in a series of books ([8], [7], [10]) and other papers. A few examples of such estimates are:

- $k_0(l^1) \leq 8$, $k_0(L^1) \leq 8$, see [1], [9],
- $k_0(H) \leq 28.99$, where H stands for an infinitely dimensional Hilbert space, see [2],
- $k_0(c_0) \leq 4\left(2+\sqrt{3}\right) \leq 14.92\ldots$, $k_0(C[0,1]) \leq 4\left(2+\sqrt{3}\right) = 14.92\ldots$. The same holds for all, the so-called extremal cut-invariant, subspaces $B(K)$ of bounded functions on an infinite set K, see [11].
- $k_0(l^\infty) \leq 12 + 2\sqrt{30} = 22.95\ldots$, see [4],
- $k_0(C_0[0,1]) \leq 2\left(2+\sqrt{2}\right) = 6.83\ldots$, where $C_0([0,1])$ is the space of continuous functions vanishing at 0, see [12]. This is, by now, the smallest known estimate regarding all Banach spaces.

In spite of efforts, the interesting case of a Hilbert H space is resistant to improvements. P. Chaoha, K. Goebel and I. Termwittupong [5] proposed a certain approach which seems to be designed only to this case. Let H be a Hilbert space and be e an arbitrary unit vector, $\|e\| = 1$. For each $a \in [-1, 1]$ define the *spherical cap*,

$$S_a = [x \in S : (x, e) \geq a].$$

Obviously, S_1 consists of exactly one point e and S_{-1} is the whole sphere S. All the caps are lipschitzian retracts of the ball B. Except the case $S_{-1} = S$, the proof of it is elementary. Define the optimal Lipschitz constant function $\kappa : [-1, 1] \to [0, +\infty)$ as

$$\kappa(a) = \inf\left[k : \text{ there exists a } k\text{-lipschitzian retraction } R : B \to S_a\right].$$

Obviously, $\kappa(1) = 0$ and $\kappa(-1) = k_0(H) \leq 28.99$. The basic results in [5] are some estimates of $\kappa(a)$ for H of finite or infinite dimension and two conclusions,

- if $\dim H < \infty$, then $\lim_{a \to -1} \kappa(a) = \infty$,
- if $\dim H = \infty$, then $\kappa(a)$ is bounded and there exists $\overline{a} > -1$ such that for all $a \in [-1, \overline{a}], \kappa(a) \geq \kappa(-1) = k_0(H)$.

Some estimates for $\overline{a}$ are also given. Analytically, in terms of the Lipschitz constant, retracting the ball onto big spherical caps is as difficult as retracting on the whole ball. The result has been improved recently in [6]. The authors proved that $\kappa(a) = \mathit{const} = k_0(H)$ on $[-1, \overline{a}]$.

Remark 1. The approach presented above is hardly transferable to general Banach spaces other than Hilbert ones. The spherical caps generated by different functionals have a variety of irregular shapes and sizes. There are no convenient analytical tools, as the inner product in a Hilbert space, to make the estimations.

Nevertheless, some tries are possible for some spaces and special selections of functionals. Next sections contain an illustration of such situation.

2. Radial mapping vs truncation. Consider as X one of the infinitely dimensional spaces c_0, c, l^∞. Let us list some facts which will be used as tools.

Fact 1:

For any Banach space X there is the so-called radial projection $P : X \setminus \{0\} \to S$ defined as

$$Px = \frac{x}{\|x\|}.$$

For any $x \neq 0$, $y \neq 0$, we have

$$\|Px - Py\| \leq \frac{2}{\max\{\|x\|, \|y\|\}} \|x - y\|. \tag{2.1}$$

Especially if for $r > 0$, $\|x\| \geq r$, $\|y\| \geq r$, then

$$\|Px - Py\| \leq \frac{2}{r} \|x - y\|. \tag{2.2}$$

For our purpose we shall often consider P restricted to the region

$$D_r = [x : r \leq \|x\| \leq 1],$$

$r < 1$. In terms of the Lipschitz condition, the mapping P considered on D_r satisfies $k(P) \leq \frac{2}{r}$. In our spaces all the above estimates are sharp. To see it, check for $r < 1$, $x = (r, r, 0, 0, 0, \dots)$ and $y = (r + \varepsilon, r - \varepsilon, 0, 0, 0, \dots)$. In some more regular spaces the constant $\frac{2}{r}$ can be replaced by a smaller one. For a Hilbert space it is $\frac{1}{r}$.

Combining the radial projection with the identity on the ball B, we get the retraction $\overline{P} : X \to B$:

$$\overline{P}x = \begin{cases} \frac{x}{\|x\|} & \text{if } \|x\| > 1 \\ x & \text{if } \|x\| \leq 1 \end{cases} = \begin{cases} Px & \text{if } \|x\| > 1 \\ x & \text{if } \|x\| \leq 1. \end{cases}$$

For our spaces $k\big(\overline{P}\big) = 2$.

The same observations are valid for any unit ball $B(a, 1)$ centered in a by shifting P to $P^a x = P(x - a) + a$ and D_r to $D_r^a = D_r + a$.

Fact 2:

In the spaces under our consideration there is another natural retraction $T : X \to B$. This is the so-called *truncation mapping*. Let

$$\alpha(t) = \max[-1, \min[1, t]] = \begin{cases} -1 & \text{for } t < -1 \\ t & \text{for } -1 \leq t \leq 1 \\ 1 & \text{for } t > 1. \end{cases}$$

Truncation onto the unit ball is defined as

$$T(x) = T(x_1, x_2, x_3, \dots) = \alpha(x) = (\alpha(x_1), \alpha(x_2), \alpha(x_3), \dots).$$

Truncation is nonexpansive, meaning $k(T) = 1$. In terms of the Lipschitz condition it is more regular than $\overline{P}$, since $1 = k(T) < k(\overline{P}) = 2$. Consequently, for any ball $B(z, r)$ we have a nonexpansive retraction $T_{(r,z)} : X \to B(z, r)$ defined as $T_{(r,z)}(x) = \alpha\left(\frac{x-z}{r}\right) + z$.

Fact 3:

In our setting the radial projection P maps D_r onto the sphere S having the Lipschitz constant $k(P) = \frac{2}{r}$. Under concern, there may be other retractions $R : D_r \to S$ satisfying the Lipschitz condition. Let $k(R) = k$. For any $\varepsilon < 1 - r$ and $p \geq 2$, consider two points in D_r:

$$x = (x_i) = \begin{cases} 1 - \varepsilon & \text{for } i \leq p \\ 0 & \text{for } i > p, \end{cases} \qquad y = (y_i) = \begin{cases} 1 & \text{for } i \leq p \\ 0 & \text{for } i > p. \end{cases}$$

Since $y \in S$, $Ry = y$. Let $Rx = (u_1, u_2, u_3, \dots)$. For at least one index j, there must be $|u_j| = 1$. If $j > p$, we have

$$1 \leq \|Rx - Ry\| \leq k\,\|x - y\| = k\varepsilon$$

and a contradiction for small ε. Thus we must have $j \leq p$. Take the point $z = (1, 1, \dots, 1, 1 - 2\varepsilon, 1, 1, \dots, 1, 0, 0, \dots)$ when $1 - 2\varepsilon$ appears on the place indexed j and the last 1 on the place p. Then we have $Rz = z$ and

$$2\varepsilon \leq \|x - Rz\| \leq k\,\|x - z\| = k\varepsilon$$

implying $k \geq 2$.

Similarly as in **Fact 1** any such R can be extended to the complement of B by putting

$$\overline{R}x = \begin{cases} Rx & \text{for } x \in D_r \\ Tx & \text{for } \|x\| > 1. \end{cases}$$

For such extension we have $k(\overline{R}) = k(R)$.

Hence, for any lipschitzian retraction $R : D_r \to S$, $k(R) \geq 2$. The natural question appears: does there exist a retraction $R : D_r \to S$ with $k(R) < k(\overline{P}) = \frac{2}{r}$?

Fact 4:

The answer is affirmative. For any $x \in D_r$ consider two radially colinear with x points $Px = \frac{x}{\|x\|}$ and $\frac{x}{r}$. Define the retraction $U : D_r \to S$ as

$$\begin{aligned} Ux &= T\left(\|x\|\,Px + (1 - \|x\|)\,\frac{x}{r}\right) = T\left(x + (1 - \|x\|)\,\frac{x}{r}\right) \\ &= T\left(\frac{1 - \|x\| + r}{r}\,x\right). \end{aligned}$$

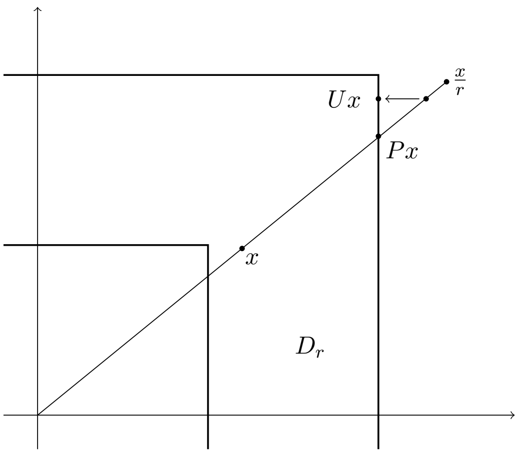

FIGURE 1. Retraction U

Now let $x, y \in D_r$ and without loss of generality assume that $\|x\| \geq \|y\|$. We have

$$\begin{aligned}
\|Ux - Uy\| &= \left\| T\left(\frac{1 - \|x\| + r}{r}\, x\right) - T\left(\frac{1 - \|y\| + r}{r}\, y\right) \right\| \\
&\leq \frac{1}{r} \left\| (1 - \|x\| + r)\, x - (1 - \|y\| + r)\, y \right\| \\
&\leq \frac{1}{r}((1 - \|x\| + r)\, \|x - y\| + \|y\| \, |\|x\| - \|y\||) \\
&\leq \frac{1}{r}\left(1 - (\|x\| - \|y\|) + r\right) \|x - y\| \leq \left(1 + \frac{1}{r}\right) \|x - y\|
\end{aligned}$$

implying $k(U) \leq 1 + \frac{1}{r} < \frac{2}{r}$.

3. Spherical caps and retractions. As declared at the beginning, we shall modify the notion of spherical caps and try to estimate the optimal Lipschitz constant of corresponding retractions. The space X, one of c_0, c, l^∞, shares a property useful for our consideration. Each is isometric to its product with the real line $\mathbf{R} \times X$ with the maximum norm. The unit ball B in X is isometric to $[-1, 1] \times B$. Consequently, B is isometric to $[0, 2] \times B$ which is also identified in $\mathbf{R} \times X$ as the unit ball centered in $(1, 0)$, namely $B((1, 0), 1)$.

Since our spaces have big flat spots on the bottom and the top of the sphere, we shall consider three types of spherical caps.

1. Flat, bottom caps:

These are the caps lying on the lower face of $B((1, 0), 1)$ parametrized by the radius $r \in (0, 1]$:

$$S_{0,r} = \{0\} \times rB.$$

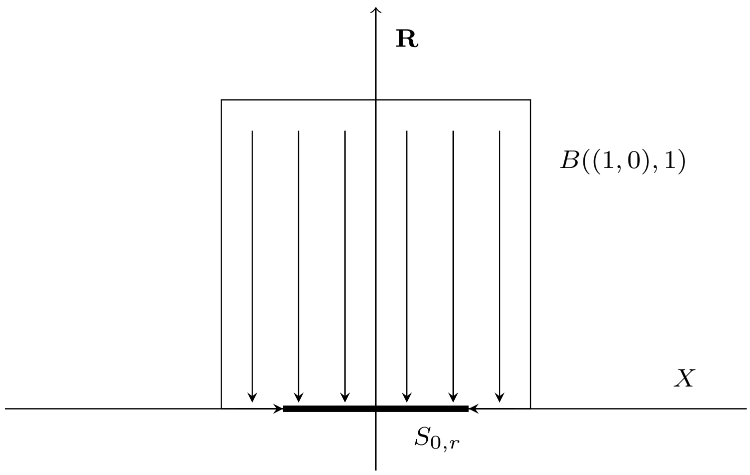

FIGURE 2. Retraction on $S_{0,r}$

This is a trivial case. The whole ball $B((1,0),1)$ can be easily retracted on such cap by the nonexpansive mapping R_r, $k(R_r) = 1$,

$$R_r(t,x) = (0, T_{(r,0)}(x)).$$

For $r = 1$ this is just the vertical projection

$$R_1(t,x) = (0,x).$$

2. Boxes of various height:

These are the caps parametrized by the height $h \in [0,2]$ defined as

$$S_{1,h} = (\{0\} \times B) \cup ([0,h] \times S).$$

Showing that $S_{1,h}$ is the retract of $B((1,0),1)$ can be done in several ways. Observe first that the construction presented in **Fact 3** of the previous section can be applied to retractions on any $S_{1,h}$. Thus for any lipschitzian retraction $R : B((1,0),1) \to S_{1,h}$, $k(R) \geq 2$. This shows a qualitative difference between retractions on bottom caps and boxes.

First consider the case $0 \leq h \leq 1$. Define the function φ and the mapping $F_h : B((1,0),1) \to B((1,0),1)$ as

$$\varphi(x) = \max\left[0, \|x\| - 1 + h\right],$$

$$F_h(t,x) = \begin{cases} (\varphi(x), x) & \text{if } t \geq \varphi(x) \\ (t,x) & \text{if } t \leq \varphi(x). \end{cases}$$

For each h, the Lipschitz constant of F_h equals one, $k(F_h) = 1$ and F_h retracts $B((1,0),1)$ onto the set

$$C_h = [(t,x) \in B((1,0),1) : t \leq \varphi(x)] = F_h(B((1,0),1)).$$

Also, C_h satisfies the two following properties. For each $(t,x) \in C_h$,

$$\|(t,x) - (1,0)\| \geq 1 - \frac{h}{2}.$$

The retraction U_h defined in **Fact 4**, on the set $D^{(1,0)}_{1-\frac{h}{2}}$ for the ball $B((1,0),1)$ retracts C_h onto $S_{1,h}$. Since

$$k(U_h) \leq 1 + \frac{1}{1-\frac{h}{2}} = \frac{4-h}{2-h},$$

composing U_h with F_h, we get the retraction $R_h = U_h \circ F_h : B((1,0),1) \to S_{1,h}$ with

$$2 \leq k(R_h) \leq \frac{4-h}{2-h}.$$

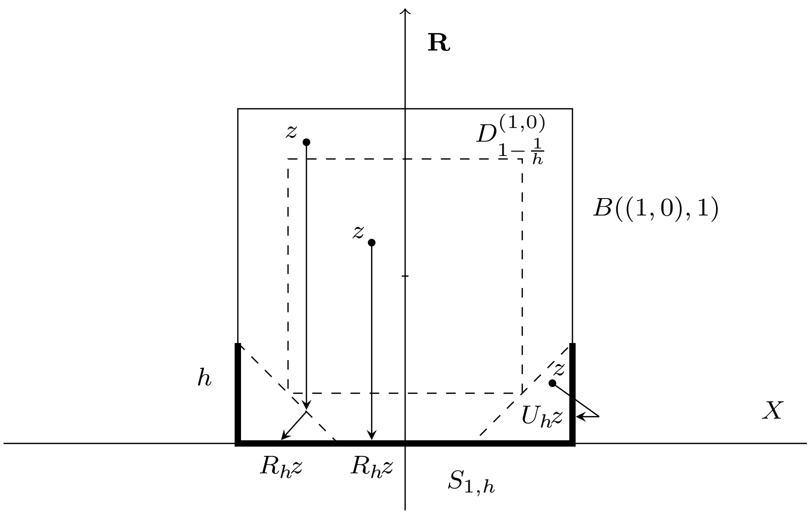

FIGURE 3. Retraction on $S_{1,h}$ for $0 \leq h \leq 1$

Especially, we get $\lim_{h\to 0} k(R_h) = 2$ and for the retraction on the half of the sphere $S_{1,1}$, $k(R_1) \leq 3$.

Now consider the case $h \in [1,2]$. It is easy to observe that the retraction R_1 can be used to construct R_h.

For any $(t,x) \in B((1,0),1)$, put $A_h(t,x) = \left(\frac{t}{h}, x\right)$ and $A_h^{-1}(t,x) = (ht, x)$ if $t \leq \frac{2}{h}$. Observe that $k(A_h) = 1$ and $k\left(A_h^{-1}\right) = h$. The retraction $R_h : B((1,0),1) \to S_{1,h}$ defined as

$$R_h(t,x) = A_h^{-1} \circ R_1 \circ A_h(t,x)$$

satisfies

$$k(R_h) \leq hk(R_1) \leq 3h.$$

Especially for the maximal box, $h = 2$, we get $k(R_2) \leq 6$.

Remark 2. The above trick can be used in slightly more general case. For any $a, r > 0$, $r \leq h$ consider a "column" set

$$K_{r,h} = [(t,x) : a \leq t \leq a+h,\ \|x\| \leq r] .$$

Then there exists a retraction V of $K_{r,h}$ onto corresponding

$$S_{r,h} = [(t,x) : t = a,\ \|x\| \leq r] \cup [(t,x) : a \leq r \leq h,\ \|x\| = r]$$

with $k(V) \leq 3\frac{h}{r}$.

3. Closing the sphere caps:

These are the caps of the form

$$S_{2,r} = \overline{S((0,1),1) \setminus (\{2\} \times rB)}$$

with $r \in (0,1]$. For $r = 0$ it is not the cap but the punctured sphere $S((0,1),1) \setminus \{(2,0)\}$ and for $r = 1$ it is just the maximal box $S_{1,2}$. Again, there are several ways for defining a retraction $R_{2,r}$ of $B((0,1),1)$ onto $S_{2,r}$. Here is one with relatively easy estimate of the Lipschitz constant.

For any $r \in (0,1]$ and $a = 1-r$ the ball $B((0,1),1)$ contains the column set

$$K_{r,1+r} = [(t,x) : 1-r \leq t \leq 2,\ \|x\| \leq r] . \tag{3.1}$$

Following **Remark 2**, there exists a retraction V of $K_{r,1+r}$ onto

$$\begin{aligned} V(K_{r,1+r}) &= S_{r,1+r} \\ &= [(t,x) : t = 1-r, \|x\| \leq r] \cup [(t,x) : 1-r \leq t \leq 2, \|x\| = r] \end{aligned}$$

having the Lipschitz constant $k(V) \leq 3\frac{1+r}{r}$.

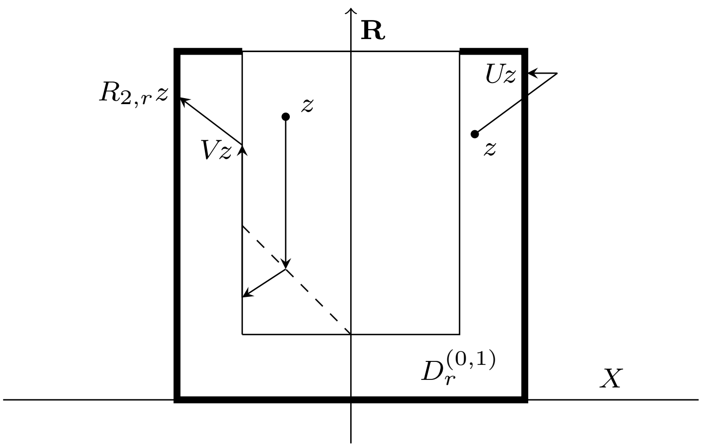

FIGURE 4. Retraction on $S_{2,r}$

Since $V(K_{r,1+r}) \subset D_r^{(1,0)}$ in $B((0,1),1)$, composing V with the retraction $U : D_r^{(1,0)} \to S((0,1),1)$, we get the retraction $R_{2,r} = U \circ V : B((0,1),1) \to S_{2,r}$ with

$$k(R_{2,r}) \leq 3\left(1 + \frac{1}{r}\right)^2. \tag{3.2}$$

Except for the flat bottom caps, it is not clear whether our constructions are the best possible and estimates of their Lipschitz constants are sharp. Especially the estimate (3.2) tends to 12 as $r \to 1$ while for $r = 1$ we have $k(R_2) \leq 6$. The situation is similar as this between maximal flat bottom cap and boxes of small height. The estimate doubles. Is this always so for any retractions or thus caused by our construction?

4. Estimates related to $k_0(X)$. The main reason of doing such constructions and estimates is finding possible retractions of the ball onto sphere and evaluations of the optimal retraction constant $k_0(X)$. The first easy observation is that the optimal Lipschitz constants for retractions of B onto $S_{2,r}$ are bounded.

Fact 5:

For any $r \in [0,1]$ and the family of all retractions $R : [0,1] \times B \to S_{2,r}$,

$$\inf[k(R) : R : [0,1] \times B \to S_{2,r}] \leq k_0^2(X).$$

To prove it, take two retractions. First, $V_1 : B((0,1),1) \to S((0,1),1)$ having $k(V_1) \leq k_0(X) + \varepsilon$. Second, $V_2 : \{2\} \times B(0,r) \to \{2\} \times S(0,r)$ with $k(V_2) \leq k_0(X) + \varepsilon$. The composition

$$R = V_2 \circ V_1 : B((0,1),1) \to S_{2,r} = \overline{S((0,1),1) \setminus (\{2\} \times rB)}$$

is a retraction with $k(R) \leq (k_0(X) + \varepsilon)^2$.

The above estimate seems to be very imprecise. One can expect that it should be $k(R) \leq k_0(X)$. However, we also have the following.

Fact 6:

There exists $a > 0$ such that for any $r < a$ and any lipschitzian retraction $R : [0,1] \times B \to S_{2,r}$,

$$k(R) \geq k_0(X).$$

Let $R : B((1,0),1) \to S_{2,r}$ be a retraction. Any segment I joining the point $(2,0)$ with a point $(2,x)$, $\|x\| = r$ is mapped by R onto a curve γ lying on $S((0,1),1)$ with ends $R(2,0)$ and $R(2,x) = (2,x)$. The length $l(\gamma) \leq k(R)r$. If none of such curves reaches the "interior" of $\{0\} \times B\,(0,r)$, the image of the ball $\{2\} \times B\,(0,r)$ can be retracted onto $\{2\} \times S\,(0,r)$ with the use of truncation by the nonexpansive mapping $V(t,x) = \big(2, T_{(r,0)}(x)\big)$. Finally, the ball $\{2\} \times B(0,r)$ is retracted onto its boundary $\{2\} \times S(0,r)$ by the composition with truncation $R^\circ = V \circ R$ having $k\big(R^\circ\big) \leq k(R)$. Thus $k_0(R) \leq k(R)$.

The condition that γ is not reaching $\{0\} \times B(0,r)$ is satisfied if

$$k(R)r \leq 2 + 2(1-r) = 4 - 2r.$$

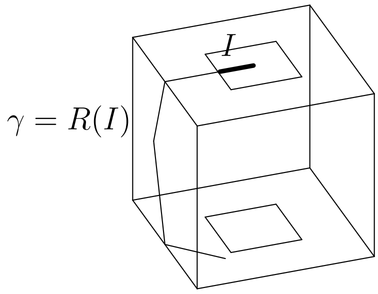

FIGURE 5. The curve $\gamma = R(I)$

Consequently, our claim holds if

$$r \leq \frac{4}{k_0(X)+2}. \tag{4.1}$$

Since for $X = c_0$ or $X = c$, $k_0(X) \leq 4\left(2+\sqrt{3}\right) = 14.9282\ldots$, then sufficient condition for (4.1) to hold is $r < 0.2362\ldots$. For $X = l^\infty$, following the estimate $k_0(l^\infty) \leq 12 + 2\sqrt{30}$, the same holds if $r < 0.1602\ldots$. The above does not show a new direct estimation for $k_0(X)$ but at least indicates that searching for it can be done via finding good retractions onto $S_{2,r}$ with sufficiently small r.

Optimization of all our estimates is a challenge.

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Kazimierz Goebel
Institute of Mathematics
Maria Curie-Skłodowska University
Pl. Marii Curie-Skłodowskiej 1
20-031 Lublin
Poland
e-mail: goebel@hektor.umcs.lublin.pl

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